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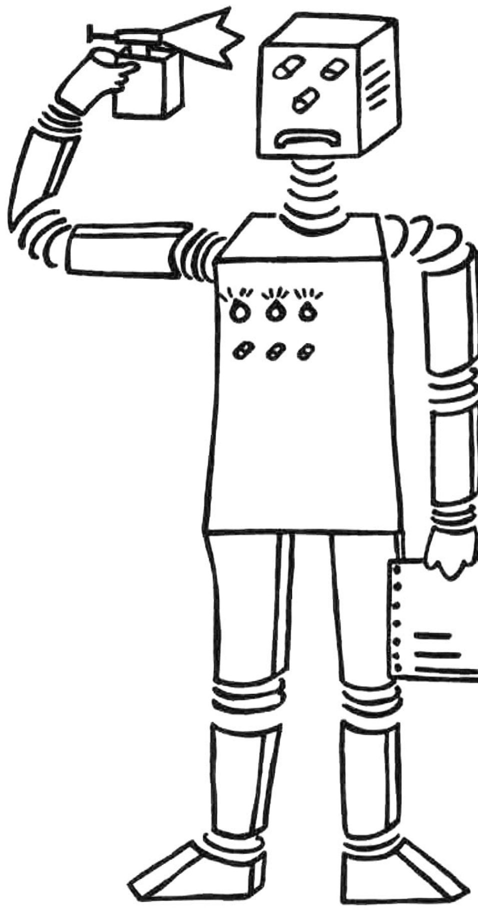

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On the Solvability of Equational Problems

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Abstract: In this report we present some results on decidability, undecidability, semi-decidability, and non-semi-decidability of general and special equational problems. Solving an equational problem is the task to find out given an equational theory and any first order formula where the equality sign is the only predicate symbol, if this formula is true in the free algebra (or in the initial algebra) of this equational theory. One is usually interested in a constructive proof of an equational problem, i.e., the assignments of existentially quantified variables with elements of this algebra have to be constructed explicitely. Special cases are equality problems, unification and disunification problems: A disunification problem is the problem of solving a system of equations and disequations (i.e. negated equations) in the free or initial algebra of an equational theory, i.e. to prove the existentially closed conjunction of the equations and disequations in this algebra. Assignments for the variables occurring in the system, that make the terms of the equations equal, but let those of the disequations different are called solutions. Unification is as common the problem of solving a system of equations only, the solutions are called unifiers. Equality problems are universally closed equations. We are only interested in the question if there is a decision or at least a semi-decision procedure for the solvability of these problems but not in the task of finding procedures that compute the solutions.
Keywords: Equational theory, equational problem, unification, disunification, decidability, semi-decidability, word problem.

1. Introduction

A well-investigated problem is the problem of solving a system of equations in a given equational theory E. This is known as E-unification and it is the task to find terms to be substituted into the variables of the system, such that the resulting equations hold with respect to the equational theory (Plotkin, 1972; Siekmann, 1978, 1989; Huet & Oppen, 1980; Fages & Huet, 1986; Kirchner, 1989). Hence a system of equations may be considered as the existential closure of a conjunction of its equations that has to be proved constructively by giving terms for these existential variables. This problem has many applications in artificial intelligence and computer science and there is a couple of papers dealing with theoretical and practical aspects of unification (cf. Siekmann, 1989, for a bibliography).

Recently the problem has been extended in the sense that the formula may be any first order formula built up by equations with conjunction, disjunction and negation. However, these general problems are investigated with restricted semantics. It makes no longer sense as for unification problems to consider solvability* with respect to all models of the theory, but for distinguished special models. The reason is that negated equations are always unsolvable with respect to all models, since equational theories always have a trivial model with one element only. Especially two cases are motivated by the applications: solvability in the free algebra of the theory and solvability in the initial algebra of the theory. These are natural extensions of the unification theoretical approach, as for unification problems it is equivalent, if they are solvable with respect to all models or the free or the initial model of a given theory (cf. Buntine & Bürckert, 1989).

Obviously it is enough to consider a prenex form of an equational problem with matrix in conjunctive-disjunctive (or disjunctive-conjunctive) normal form; moreover, most practical applications deal with a prefix having the shape $\forall \ldots \forall\exists \ldots \exists\forall \ldots \forall$, no deeper nested alternation occurs. Thus we will concentrate on this kind of equational problems, and we will speak of general equational problems whenever we address solvability of arbitrary equational formulae (or their disjunctive normal forms). As other well-known instances of equational problems we will discuss solvability of equality problems and matching problems: The first one consists of an equation with universal quantifiers only, the second one of an equation with leading universal quantifiers for the variables of one of the two terms and with existential quantifiers for the variables of the other term.

It is well-known that solvability of each of these instances is an undecidable problem – solvability of equality problems is also known as the word problem for the given equational theory. It is also well-known that unification problems and word problems are semi-decidable. H. Comon (1988) shows that equational problems are no longer semi-decidable, when disequations are allowed. The reason is that equational problems with disequations only are equivalent to negations of equality problems and hence cannot be semi-decidable. We will investigate decidability and semi-decidability of some other special equational problems as well as of the general one and still more interesting also some relative decidability questions, for instance, decidability of E-unification for theories with decidable word problem.

The paper is organized as follows. In the next section we recall the definitions of terms, substitutions, equations, disequations, and equational theories, in order to have the prerequisites for our definition of equational problems, several instances, and their solvability notion in section 3. The main section contains the decidability and undecidability results for solvability of these problems and finally in section 5 we give a ground solvability result and some application of this.

2. Terms, Substitutions, Equations, Disequations

We assume the reader to be familiar with the common notions of universal algebra and equational logic as for example can be found in (Burris & Sankappanavar, 1981; Grätzer, 1979;

* We are going to stress the notion of solvability a little bit in this generalization. In fact it means provability in our context. Usually one would only speak of solvability if the equational problem is starting with existential quantifiers.

see also Taylor, 1979). Our notions are consistent with those usual in unification theory (Kirchner, 1989; Siekmann, 1989). We just recall the notations of term algebras, substitutions and equational theories as we will need them in the further sections.

Given a *signature* Σ of function symbols with fixed arities and an infinite set $\mathcal{V}$ of variables we denote the *Σ-term algebra over* $\mathcal{V}$ by $\mathcal{T}_\Sigma(\mathcal{V})$. Sometimes we are only interested in the Σ-term algebra $\mathcal{T}_\Sigma(V)$ over a subset V of $\mathcal{V}$; especially, if V is the empty set the elements of $\mathcal{T}_\Sigma(\emptyset)$ are also called *ground Σ-terms*. The set of variables occurring in any syntactical object O built up by terms is denoted by $Var(O)$. A *Σ-substitution* σ *over* $\mathcal{V}$ is an endomorphism of $\mathcal{T}_\Sigma(\mathcal{V})$ that moves at most finitely many variables. We denote the set of all Σ-substitutions by SUB_Σ. The finite set of variables $Dom\sigma = \{x \in \mathcal{V}\colon \sigma x \neq x\}$ that are moved by a substitution σ is called the *domain*, the set of their images $Cod\sigma = \{\sigma x\colon x \in Dom\sigma\}$ is called the *codomain* of σ; $VCod\sigma = Var(Cod\sigma)$ denotes the set of variables introduced by σ. A *ground* substitution is a substitution γ with $VCod\gamma = \emptyset$. In the following we drop the index, when we assume the signature Σ to be fixed and clear from the context.

Pairs of terms $s = t$ and $p \neq q$ are called *Σ-equations* and *Σ-disequations*, respectively. A set of equations E induces a congruence relation $=_E$ on the term algebra called an *equational theory over Σ* – or a theory, for short – by the following inductive closure construction:

if $s = t \in E$ and $\sigma \in SUB$ then $\sigma s =_E \sigma t$

for all $t \in \mathcal{T}(\mathcal{V})$: $t =_E t$

if $s =_E t$ then $t =_E s$

if $r =_E s$ and $s =_E t$ then $r =_E t$

if $s_1 =_E t_1, \ldots, s_n =_E t_n$ and $f \in \Sigma$ with arity $n \geq 1$ then $f(s_1,\ldots,s_n) =_E f(t_1,\ldots,t_n)$.

We frequently denote the generating set E itself as (equational) theory, when we address the congruence $=_E$. Then we call two terms s and t E-equal, when $s =_E t$, and we denote the fact that two terms p and q are not E-equal by $p \neq_E q$. Notice, that a theory always comes with a generating set E *and* a signature Σ containing at least the function symbols of the terms in E (function symbols of Σ not occurring in the terms of E are called *free*).

By Birkhoff's completeness theorem of equational logic an equational theory is the set of all equations that are true in all models of its generating set. Tarski has shown that the term algebra over countably infinitely many variables is generic for equational theories, i.e., the equations of an equational theory are exactly those that are true in the E-quotient of the term algebra $\mathcal{T}(\mathcal{V})/{=_E}$, also sometimes called the *E-free algebra*.

3. Equational Problems

Equational logic in the above sense is traditionally concerned with universally quantified (equational) formulae: Both the equations in the generating set E – considered as axioms – as well as the derived equations that form the equational theory are universally quantified (atomic) formulae. In the sequel we will leave this way and follow the unification theoretical view of proving existentially quantified formulae to be consequences of the axioms in a generating set of an equational theory. If the formulae that are to be proved consequences of the theory are (existentially quantified conjunctions of) equations only then it is equivalent to prove them with respect to all models of an equational theory or with respect to the free algebra of the theory (cf. Buntine & Bürckert, 1989). However, if we drop this restriction and allow arbitrary formulae to be proved consequences of a theory then, we have to distinguish the models we are interested in. The reason is that disequations (negated equations) cannot be true with respect to all models of an equational theory as they will always be false in the trivial (one-element) model. In this paper we will consider mainly the case where the intended model is the E-free algebra, only in the last section we will also look at the initial model of an equational theory, that is the E-quotient of the ground term algebra. These two are the models where equational problem solving has focussed on.

A *general equational problem* $\langle Q\,.\Phi \rangle_E$ is given by a theory E, a finite set Φ of systems $\Gamma_i \cup \Delta_i\, (1 \le i \le n)$ of equations and disequations (considered as a disjunction of the sets $\Gamma_i \cup \Delta_i$, which are in turn considered as conjunctions of the equations and disequations), and any quantifier prefix Q binding all variables in Φ. A *(special) equational problem* is a general equational problem with a quantifier prefix $\forall_U\, \exists_X\, \forall_Y$, where U, X, and Y are disjoint sets or lists of variables in $Var(\Phi)$. The existentially quantified variables in X are the *unknowns* and the universally quantified variables in U and Y are the *independent* and the *dependent parameters*, respectively. If the set of dependent parameters Y is empty, we call the problem an *E-disunification* problem, and if in addition the systems of disequations Δ_i are empty (and there is only one system Γ of equations) we call it an *X-restricted E-unification* problem. If in this case $Var(s) \subseteq U$ for all $s = t$ in Γ, then the problem is also known as an *E-matching* problem of the terms t to the terms s. If U is also empty, we have the common *E-unification* problem. Its counterpart, an equational problem with empty sets U, Y, Γ, i.e., without equations and parameters, and only one system Δ of disequations is called a proper E-disunification problem (cf. table 1).

(special) equational problem	$\langle \forall_U\, \exists_X\, \forall_Y\, .\, \Phi \rangle_E$	$U \cup X \cup Y = Var(\Phi)$
equality	$\langle \forall_U\, .\, s = t \rangle_E$	$U = Var(s,t)$
unification	$\langle \exists_X\, .\, \Gamma \rangle_E$	$X = Var(\Gamma)$
matching	$\langle \forall_U\, \exists_X\, .\, s_i = t_i : 1 \le i \le n \rangle_E$	$U = Var(s_i), X = Var(t_i) \setminus Var(s_i)$
restricted unification	$\langle \forall_U\, \exists_X\, .\, \Gamma \rangle_E$	$U \cup X = Var(\Gamma)$
parametrized unification	$\langle \forall_U\, \exists_X\, \forall_Y.\, \Gamma \rangle_E$	$U \cup X \cup Y = Var(\Gamma)$
proper disunification	$\langle \exists_X\, .\, \Delta \rangle_E$	$X = Var(\Delta)$
disunification	$\langle \exists_X\, .\, \Gamma \cup \Delta \rangle_E$	$X = Var(\Gamma, \Delta)$
parametrized disunification	$\langle \forall_U\, \exists_X\, \forall_Y.\, \Gamma \cup \Delta \rangle_E$	$U \cup X \cup Y = Var(\Gamma, \Delta)$

Table 1: Equational Problems

A general equational problem is called *solvable* iff the E-free algebra $\mathcal{T}(\mathcal{V})/{=_E}$ is a model of the corresponding formula. In case of special equational problems this is equivalent to testing if for every substitution α with $Dom\alpha = U$, $VCod\alpha \cap (U \cup X \cup Y) = \emptyset$ there exists a substitution σ

with $Dom\sigma = X$, $VCod\sigma \cap (X \cup Y) = \emptyset$, such that for all substitutions δ with $Dom\delta = Y$, $VCod\delta \cap Y = \emptyset$ we have:

$\delta\sigma\alpha s =_E \delta\sigma\alpha t$ for all $s = t$ in Γ_i and $\delta\sigma\alpha p \neq_E \delta\sigma\alpha q$ for all $p \neq q$ in Δ_i (for some i, $1 \leq i \leq n$).

Notice, that by the restrictions on the domains and codomains of the substitutions we have that the involved substitutions commute: $\delta\sigma = \sigma\delta$ etc. If there are no independent parameters, i.e., if $U = \emptyset$, then the substitutions σ are called *solutions* of the problem, in the other case the notion of solution is not quite adequate, since they would depend on the chosen α. However, if an equational problem contains no disequations (we sometimes speak of a parametrized [restricted] unification problem in this case), the dependency of the substitution σ on the substitutions α need not to be treated explicitely. In these cases the problem is solvable iff there is a substitution σ with $Dom\sigma = X$, $VCod\sigma \cap (X \cup Y) = \emptyset$, and $\sigma s =_E \sigma t$ for all $s = t$ in Γ_i (for some i). Notice, that such problems can be equivalently transformed in problems with empty U by replacing each variable of U by a new free constant. This is essentially a consequence of the theorem on constants (Shoenfield, 1967) and the fact that for equations it is equivalent to prove them over the E-free algebra or with respect to all models of E (cf. Buntine & Bürckert, 1989).

Our definitions are consistent with the usual definition of solvable unification problems and matching problems (cf. Siekmann, 1989). The notion of solutions also coincides with the common definition of solutions of unification and matching problems – they are called unifiers and matchers, respectively.

4. Solvability Results

In this section we will address the problem of testing solvability of equational problems, that is the question if there is a decision procedure or at least a semi-decision procedure for these problems.

Theorem 1: *a) E-equality, E-unification, E-matching, and restricted E-unification are always semi-decidable with or without parameters.*
b) There exist equational theories (E, Σ), where E-equality, E-unification, E-matching, or restricted E-unification is undecidable.
c) There exist equational theories (E, Σ), where proper E-disunification and hence arbitrary equational problems are not even semi-decidable.

Proof: a) Semi-decidability of E-equality is well-known. Semi-decidability of the other problems follows with recursive enumerability of the respectively restricted substitutions and with semi-decidability of E-equality by a "zip" argument (we ssume that independent parameters are replaced by new free constants):

Enumerate all substitutions and test E-equality of σs and σt
(enumeration of substitutions and E-equality test procedure have to run alternately).
Stop, when E-equality test answers yes for some of the enumerated substitutions.

Obviously, the procedure terminates, when a solution has been found.

b) Undecidability of E-equality is again well-known (see for example Taylor, 1979). Siekmann & Szabo (1987) show that unification in the theory *DA* of associativity and distributivity of two binary function symbols is undecidable. We will prove undecidability of restricted E-unification and E-matching in a more general case (see Theorem 2 below).

c) Negation of proper E-disunification is equivalent to E-equality, hence it cannot even be semi-decidable (cf. Comon 1988). See also Theorem 2 below.∎

Since there are many theories *E*, where algorithms and procedures have been developed computing the solutions of E-unification problems (or a representation), and since other equational problems are related in the sense, that they often can be reduced to E-unification, one may ask, if we have better results for theories with decidable E-unification. The answer is unfortunately negative. The technique of the following proof can also be used, in order to prove that the introduction of sorts (sort-unification) can make unification undecidable (cf. Schmidt-Schauß, 1986).

Theorem 2: *There exist equational theories (E, Σ), such that E-unification is decidable, however, E-equality, E-matching, or restricted E-unification is undecidable, and proper E-disunification and hence arbitrary equational problems are not even semi-decidable.*

Proof: 1. Take any "collapse free" theory (E, Σ) (i.e., without "collapse" equations $t = x$, where x is a variable and t is not) with undecidable word problem; for example a semigroup with undecidable word problem (cf. Markov, 1947; Post, 1947). We construct a new theory (E', Σ') by extending every n-ary function f in Σ $(n \geq 0)$ to an (n+1)-ary function f'; hence there is no constant now in Σ'. We transform every equation of E to an equation in E' by replacing the f by f' and fill the new (last) argument positions all with the same new variable. For example a Σ-equation $f(x, g(y, c)) = h(z)$ is transformed into the Σ'-equation $f'(x, g'(y, c'(v), v), v) = h'(z, v)$ with the new variable v. Finally we add a new constant a' to Σ' together with the following equations to E':

$$f'(x_1,\ldots,x_n,a') = a' \quad \text{for all n-ary functions } f \text{ in } \Sigma.$$

Hence every non-variable Σ'-term, whose last argument position is a', is E'-equal to a' (here we need that the theory is collapse free, otherwise E' would be inconsistent). Since this works recursively, we have

$$f'(t_1,\ldots,t_n, g'(s_1,\ldots,s_m, a')) =_{E'} f'(t_1,\ldots,t_n,a') =_{E'} a',$$

i.e., especially all ground Σ'-terms are E'-equal to a'. (Notice, that there are no constants except a').

Thus every E'-unification problem is now trivially solvable by substituting every variable of the Σ'-terms for example bythe constant a'. But, obviously it is not decidable, whether two Σ'-terms, that do not contain the constant a' are E'-equal, since otherwise E-equality were decidable: For the class of Σ'-terms having all the same fixed variable in the last argument position E'-equality is equivalent to E-equality of the corresponding Σ-terms (for this reduction we need that E contains no collapse equations).

2. Undecidability of matching and restricted unification can be proved by a similar technique (Bürckert, 1986, 1989) starting with a theory (E, Σ) (without collapse equations), where unification is undecidable (for example the theory *DA* of distributivity and associativity of two binary functions, cf. Siekmann & Szabo, 1987). We do the same transformations as above, in order to obtain a theory (E', Σ') with decidable unification, but with undecidable restricted unification. The problems, where we are not allowed to substitute the variables in the last argument position of the Σ'-terms – let these variables be collected in U – are undecidable. In order to obtain a theory (E'', Σ''), where matching is undecidable, we add a new ternary function h' and a new unary function k' to the signature Σ' generated above, and we add the following equations for every (n+1)-ary function f' of the signature Σ'' (including h' and k') to the theory E'':

$$f'(x_1,\ldots, x_n, h'(u, v, w)) = a', f'(x_1,\ldots, x_n, k'(v)) = a'$$

and finally the equation

$$h'(x, x, y) = k'(y).$$

Notice, that every Σ'-term is also a Σ''-term. Now, unification is still decidable by replacing the last argument position of Σ''-terms with non-variables as above, but matching is undecidable: The problems with Σ''-equations $h'(s, t, v) = k'(v)$, where s and t are Σ'-terms with the variable v in the last position of every of there subterms, are only matchable (the variable v is not allowed to be instantiated, i.e., $U = \{v\}$), when s and t are restricted E'-unifiable, where again v is in U.

3. Since negated E-equality problems are proper E-disunification problems, the latter one cannot be semi-decidable, when the first one is undecidable. Given two terms s and t let $X = Var(s, t)$.

The word problem for E is equivalent to:

Does $\delta s =_E \delta t$ hold for all substitutions δ with $Dom\delta = X$?

Its negation is:

Does there exist some δ with $Dom\delta = X$, such that $\delta s \neq_E \delta t$?

Obviously, the negation is equivalent to the question for the existence of solutions to the disunification problem $\langle \exists_X . s \neq t \rangle_E$. Now, when E-disunification were semi-decidable, E-equality as well as its negation were semi-decidable, hence E-equality were decidable, a contradiction.■

Remarks: Parameterized unification is equivalent to a special case of unification with free function symbols (see Theorem 4 below). Restricted unification is equivalent to unification with free constants (see Bürckert, 1986, 1989). Hence the above theorem implies that unification may become undecidable, when we extend the signature of a given theory with free constants.

Corollary: *There exist theories with decidable unification for N free constants (N ≥ 0), but undecidable unification for K > N free constants.*

Proof: a) Let $N = 0$. Take the construction of the above theorem for restricted unification.

b) Let $N > 0$. We start again with some collapse free theory (E, Σ) with undecidable unification (for example DA). We construct a new theory (E', Σ') with $N-1$ new free constants $c_1,\ldots, c_{N-1}$ and with an $(n + N)$-ary function symbol f' for every n-ary function symbol f in Σ $(n \geq 0)$, and a (non-free) constant a'. We transform the equations of E analogously to the above construction, but with N new variables for the new argument positions (for instance, the equation $f(x, g(y, c)) = h(z)$ is transformed into the equation

$$f'(x,\ g'(y,\ c'(v_1,\ldots,\ v_N),\ v_1,\ldots,\ v_N\ ),\ v_1,\ldots,\ v_N\ ) = h'(z,\ v_1,\ldots,\ v_N\ )).$$

Then we add the equations

$$f'(x_1,\ldots,\ x_n,y_1,\ldots,y_{i-1},\ z,\ y_{i+1},\ldots,\ y_{j-1},\ z,\ y_{j+1},\ldots,\ y_N) = a'$$

for every n-ary function f in Σ and every pair i, j $(1 \leq i < j \leq N)$, i.e., whenever two of the new top argument positions of a term contain E'-equal terms, this term is E'-equal to a'. Finally we add the equations (for every n-ary function in Σ)

$$f'(x_1,\ldots,\ x_n,y_1,\ldots,y_{i-1},\ a',\ y_{i+1},\ldots,\ y_N) = a'$$

and

$$f'(x_1,\ldots,\ x_n,y_1,\ldots,y_{i-1},\ g'(z_1,\ldots,\ z_m),\ y_{i+1},\ldots,\ y_N) = a'$$

for every m-ary function g in Σ and every new argument position i. Hence especially all ground terms are E'-equal to the constant a': Either one of the new positions contains a term that is not one of the free constants or all these positions are occupied by free constants, but then at least two positions contain the same free constant.

In this theory (E', Σ') every two terms are trivially unifiable by replacing all variables with ground terms, for example with the constant a', but if we add a further free constant c_N, all E'-unification problems with terms, where the new positions are occupied pairwise differently by the constants $c_1,\ldots, c_N$, are equivalent to the E-unification problems and hence undecidable.■

The above constructions may be somewhat artificial for most applications, since the theories do not contain free constants or at most finitely many (of course this is not an artificial restriction, when we are interested in ground solvability only). On the other hand all known unification procedures can handle (arbitrarily many) free constants, i.e. they can solve unification problems with arbitrarily many free constants. Hence we may ask, what happens, if unification is decidable for a given theory with free constants.

Theorem 3: *Decidability of E-unification with infinitely many free constants implies decidability of E-equality, E-matching, restricted E-unification, proper E-disunification, and semi-decidability for special equational problems (in E) without independent parameters.*

Proof: 1. Decidability of E-equality, E-matching, and restricted E-unification are obvious, when we replace all variables that are not unknowns by new free constants (by definition these problems have only independent parameters).

2. In order to give a decision procedure for proper E-disunification problems $\langle \exists_X . \Delta \rangle_E$ we consider the negation:

For all substitutions σ in X we have $\sigma p =_E \sigma q$ for some $p \neq q$ in Δ.

Obviously, this is equivalent to the E-equality problem:

$$p =_E q \text{ for some } p \neq q \text{ in } \Delta.$$

Hence we can decide the problem by the E-equality decision procedure.

3. Finally we consider arbitrary equational problems $\langle \exists_X \forall_Y . \Gamma_i \cup \Delta_i\text{: } 1 \leq i \leq n \rangle_E$ for E without independent parameters:

There is a substitution σ with $Dom\sigma = X$, $VCod\sigma \cap (X \cup Y) = \emptyset$,
such that for all substitutions δ with $Dom\delta = Y$ we have (for some i, $1 \leq i \leq n$)
$\delta\sigma s =_E \delta\sigma t$ for all $s = t$ in Γ_i or $\delta\sigma p \neq_E \delta\sigma q$ for all $p \neq q$ in Δ_i.

A semi-decision procedure is given by:

Enumerate all substitutions σ with $Dom\sigma = X$, $VCod\sigma \cap (X \cup Y) = \emptyset$,

and for each system $\Gamma_i \cup \Delta_i$ $(1 \leq i \leq n)$
test (proper) E-disunifiability of the instantiated disequations $\sigma s \neq \sigma t$
(for all $s = t$ in Γ_i)
test Y-restricted E-unifiability of the instantiated equations $\sigma p = \sigma q$
(for $p \neq q$ in Δ_i).

If some of these tests answers **no**, stop with answer **yes** (the current σ is a solution).

The procedure terminates for solvable problems, since restricted E-unification and proper E-disunification are decidable. Notice, that E-disunifiability of $\sigma s \neq \sigma t$ or Y-restricted E-unifiability of $\sigma p = \sigma q$ falsifies

"for each δ with $Dom\delta = Y$: $\delta\sigma s =_E \delta\sigma t$ and $\delta\sigma p \neq_E \delta\sigma q$"

and hence it implies unsolvability of the equational problem.■

Remark: Notice, that if general equational problems are also semi-decidable, then they must be decidable, since the negation of an equational problem is again an equational problem.

Proof part 2 is based on the equivalence of E-equality problems and the negation of proper E-disunification problems. This equivalence implies the following proposition.

Proposition: *Proper E-disunification is decidable, iff E-equality is decidable.*

We close this section with a result addressing the relationship between E-unification with free functions and parametrized E-unification.

Theorem 4: *Decidability of E-unification with free function symbols implies decidability of parametrized E-unification.*

Proof: We show that every parametrized E-unification problem is equivalent to an E-unification with additional new free function symbols. This looks like a dual construction of

Skolemization. In fact, if we remember that E-unifiability of a system of equations is equivalent to proving the corresponding formula to be a consequence of the equational theory E, which is in turn equivalent to unsatisfiability of the set of formulae containing the formulae of the equational theory and the negation of the unification formula, then it is exactly the same as Skolemization.

Let Γ be a set of equations with variables in $Var(\Gamma) := \{x_i, y_j: 1 \le i \le n, 1 \le j \le m\}$, and let $\lambda := \{y_j \leftarrow f_j(x_1,\ldots,x_n): 1 \le j \le m\}$ be a substitution, where f_j are n-ary free function symbols not occurring in the equation system Γ. Then

$$(*)\quad \sigma \text{ solves } \langle \exists x_1,\ldots,x_n\ \forall y_1,\ldots,y_m\ .\ \Gamma \rangle_E \quad \text{iff} \quad \sigma \text{ solves } \langle \exists x_1,\ldots,x_n\ .\ \lambda\Gamma \rangle_E.$$

Let σ be a substitution with $Dom\sigma \cap \{y_1,\ldots,y_m\} = \emptyset$, $VCod\sigma \cap \{y_1,\ldots,y_m\} = \emptyset$. Then we have by definition (with $\lambda := \{y_j \leftarrow f_j(x_1,\ldots,x_n): 1 \le j \le m\}$):

(1) σ solves $\langle \exists x_1,\ldots,x_n\ \forall y_1,\ldots,y_m\ .\ \Gamma \rangle_E$ iff $\sigma s =_E \sigma t$ (for all $s = t$ in Γ)

(2) σ solves $\langle \exists x_1,\ldots,x_n\ .\ \lambda\Gamma \rangle_E$ iff $\sigma\lambda s =_E \sigma\lambda t$ (for all $s = t$ in Γ).

Because of the disjointness requirements we have that $\sigma\lambda = \lambda_\sigma\sigma$ with the substitution $\lambda_\sigma := \{y_j \leftarrow f_j(\sigma x_1,\ldots,\sigma x_n): 1 \le j \le m\}$. Hence we just have to show that

$$\sigma s =_E \sigma t \quad \text{iff} \quad \lambda_\sigma\sigma s =_E \lambda_\sigma\sigma t \quad (\text{for all } s = t \text{ in } \Gamma).$$

One direction ("$\Rightarrow$") is trivial, since $\lambda_\sigma\sigma$ is an instance of σ. For the other direction ("$\Leftarrow$") $\sigma\Gamma$ can be considered as a "variable abstraction" of $\lambda_\sigma\sigma\Gamma$; adding new free functions f_j is just a combination of the given equational theory E with the empty theory E' for these fucntion symbols f_j. Hence this direction follows with Lemma 3.2.1.4 of Tiden (1986); see also (Schmidt-Schauß, 1987):

We get $\sigma\Gamma$ by an abstraction of the "E-alien" terms $f_j(\sigma x_1,\ldots,\sigma x_n)$ in $\lambda_\sigma\sigma\Gamma$. Hence with Tidens Lemma 3.2.1.4 we have $\sigma s =_E \sigma t$ iff $\lambda_\sigma\sigma s =_{E\cup E'} \lambda_\sigma\sigma t$ (for all $s = t$ in Γ).

Now, (*) immediately implies the theorem.■

5. A Ground Solvability Problem

Finally we want to address decidability of the existence of ground solutions of certain equational problems. An equational problem $\langle \forall_U\ \exists_X\ \forall_Y\ .\ \Phi \rangle_E$ is called *ground solvable*, iff for all ground substitutions α with $Dom\alpha = U$ there is a ground substitution σ with $Dom\sigma = X$, such that for each ground substitution δ with $Dom\delta = Y$ we have (for some i, $1 \le i \le n$):

$$\delta\sigma\alpha s =_E \delta\sigma\alpha t \text{ for all } s = t \text{ in } \Gamma_i \text{ and } \delta\sigma\alpha p \ne_E \delta\sigma\alpha q \text{ for all } p \ne q \text{ in } \Delta_i.$$

Theorem 5: *Decidability of E-unification implies semi-decidability of ground solvability of proper E-disunification problems (without independent parameters), i.e., of the problems*

$$\langle \exists_X\ \forall_Y\ .\ p_1 \ne q_1,\ldots, p_m \ne q_m \rangle_E.$$

Proof: A semi-decision procedure is given by:

Enumerate all ground substitutions σ with $Dom\sigma = X$, and test unifiability of $\sigma p_i = \sigma q_i$ for each i.

If it is not unifiable for some i, then σ is a solution. Otherwise $\delta\sigma p_i \neq_E \delta\sigma q_i$ for all i and for all δ. Hence σ cannot be a solution.■

Comon (1986), Kirchner & Lescanne (1987), Comon & Lescanne (1988) address an application of solving equational problems in the initial algebra, namely, sufficient completeness of specifications of functions by a term rewriting system. The above result implies that such problems are decidable (also in the more general case of rewrite systems modulo an equational theory), whenever unification in this theory is decidable.

Corollary: *Sufficient completeness of the specification of a function f in (R, E, Σ) by a rewrite system R modulo an equational theory (E, Σ) is decidable, when E-unification is decidable.*

Proof: Sufficient completeness of the specification of an n-ary function f in $(R, E, \Sigma), f \notin \Sigma$, is equivalent to ground solvability of the following X-restricted E-unification problem of alternative equations:

$$\langle\ \forall_X \exists_Y\ .\, f(x_1,\ldots,x_n) = l_1 \vee \ldots \vee f(x_1,\ldots,x_n) = l_k \rangle_E$$

where the l_j are the left hand sides of the rewrite rules in R and Y is the set of variables occurring in these rules, while $X := \{x_1,\ldots,x_n\}$ are new variables (cf. Comon, 1986; Kirchner & Lescanne, 1987; Comon & Lescanne, 1988)).
Now, we have that sufficient completeness is semi-decidable as well as its negation (Theorem 5), hence it must be decidable.■

6. Conclusion

We have proved the undecidability of some equational problems also if unification is decidable. However, when unification with free function symbols is decidable – all known unification algorithm are able to solve equations with free constants – then some special cases of equational problems are also decidable, and we claim that this is also true for general equational problems. We have shown that general equational problems are at least semi-decidable in this case, which, however, need not be the case, when unification is undecidable. This demonstrates that in general equational problem solving is a difficult task. Nevertheless, in special cases it might be quite useful as several applications have shown in the past.

For the problem of decidability of parametrized unification and decidability of general equational problems, when unification is decidable, we have not found a proof. Hence we state the following

Open Problems:

1. Can E-unification become undecidable, if we add free function symbols?

2. Are arbitrary equational problems decidable, if E-unification with free constants is decidable?

A hint for a way how to find a solution to the latter might be that the negation of a general equational problem is again a general equational problem. Hence it suffices to prove semi-decidability of general equational problems (if E-unification with free constants is decidable).

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