C*-algebras generated by two partitions of unity and derived quantum graphs

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Abstract

This thesis investigates two different topics in the field of operator algebras and noncommutative geometry. In the first part, we study bipartite graph C*-algebras, a class of C*-algebras generated by two partitions of unity with orthogonality relations prescribed by a bipartite graph. Our main result gives a classification of bipartite graph C*-algebras in terms of their space of one- and two-dimensional irreducible representations. The C*-algebras associated to the hypercubes of arbitrary dimension are studied in more detail. For these we obtain an explicit description in terms of continuous matrix-valued functions on a standard simplex.

The second part of this thesis contributes to the theory of quantum graphs, a novel generalization of classical graphs. Given a free action of a finite group on a finite classical graph Γ, a construction of Gross and Tucker describes Γ as the so-called *derived graph* of a *voltage graph*. The latter is a finite graph together with a labeling of the edges by elements of a finite group. We generalize the derived graph construction to quantum graphs and finite abelian groups, and we characterize precisely which quantum graphs can be written as derived quantum graphs. Via a choice of a group action in the construction of the derived quantum graph, a fixed voltage quantum graph can give rise to different non-isomorphic derived quantum graphs which are in this case quantum isomorphic to each other.

Zusammenfassung

Diese Dissertation beschäftigt sich mit zwei verschiedenen Themen aus dem Gebiet der Operatoralgebren und nicht-kommutativen Geometrie. Im ersten Teil untersuchen wir bipartite-Graph-C*-Algebren, eine Klasse von C*-Algebren, die von zwei Partitionen der Eins erzeugt werden, wobei ein bipartiter Graph bestimmte Orthogonalitätsbedingungen zwischen den Partitionen vorgibt. Unser Hauptresultat klassifiziert bipartite-Graph-C*-Algebren durch ihren Raum der ein- und zwei-dimensionalen irreduziblen Darstellungen. Weiter untersuchen wir die C*-Algebren der Hyperwürfel genauer und erhalten eine explizite Beschreibung durch stetige matrixwertige Funktionen auf einem Standardsimplex.

Der zweite Teil dieser Arbeit trägt zur Theorie der Quantengraphen bei. Eine klassische Konstruktion von Gross und Tucker beschreibt einen beliebigen Graphen Γ als *abgeleiteten Graph* eines sogenannten *Spannungsgraphen*, falls eine freie Wirkung einer endlichen Gruppe auf Γ gegeben ist. Ein Spannungsgraph ist dabei ein klassischer Graph zusammen mit einer Beschriftung der Kanten durch Elemente einer endlichen Gruppe. Wir verallgemeinern diese Konstruktion auf Quantengraphen und endliche abelsche Gruppen und charakterisieren genau, welche Quantengraphen als abgeleitete Quantengraphen dargestellt werden können. Durch die Wahl verschiedener Gruppenwirkungen in der Konstruktion des abgeleiteten Quantengraphen kann ein fester Quanten-Spannungsgraph verschiedene nicht-isomorphe abgeleitete Quantengraphen erzeugen, die in diesem Fall paarweise quantenisomorph sind.

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1. Introduction

This thesis is divided into two parts, of which the first deals with C*-algebras generated by two partitions of unity, and the second part introduces and investigates derived quantum graphs. What unites both topics is the notion of a *C*-algebra*. A C*-algebra is a complex algebra A together with an involution and a norm that satisfies the C*-identity

$$\|aa^*\| = \|a\|^2 \quad \text{for all } a \in A.$$

One of the earliest fundamental results about C*-algebras is *Gelfand duality*: There is a systematic correspondence between locally compact Hausdorff spaces X and commutative C*-algebras A, given by the mapping $X \mapsto A := C_0(X)$.

Gelfand duality provides a complete description of commutative C*-algebras in terms of locally compact Hausdorff spaces. In the last decades an enormous amount of research has been devoted to extending this understanding to non-commutative C*-algebras where things are much less transparent. The first part of the present thesis contributes to this by studying a particular class of C*-algebras.

Further, motivated by Gelfand duality one can interpret C*-algebras as *noncommutative topological spaces*. But not only topological spaces can be generalized to the non-commutative setting: For instance, (differential) geometry [23], group theory [29, 47, 101] and probability theory [88] have all been equipped with non-commutative generalizations. Sometimes these non-commutative versions fill a gap that has been open in the classical theory. For example, the theory of quantum groups provides a satisfactory generalization of Pontryagin duality to non-abelian groups [57, 60]. More recently, quantum graphs have been introduced as a non-commutative generalization of classical graphs. In the second part of this thesis we contribute to their study by investigating the novel notion of a derived quantum graph.

In the following sections we elaborate on both parts of this thesis in more detail.

1.1. Part I: C*-algebras generated by two partitions of unity

In the study of C*-algebras, a major aim is to understand the universe of all C*-algebras. This includes two challenges: The first is to describe C*-algebras by a concrete construction. As an example, Gelfand duality tells us that any commutative C*-algebra can be constructed as $C_0(X)$ for some locally compact Hausdorff space X. A second challenge is to distinguish C*-algebras up to isomorphism, which we call the classification problem. Again Gelfand duality provides such a classification: It says that two commutative C*-algebras $C_0(X)$ and $C_0(Y)$ are isomorphic if and only if the underlying spaces X and Y are homeomorphic.

For commutative C*-algebras, Gelfand duality solves both the construction and classification problem perfectly. What can we say about non-commutative C*-algebras? Firstly, every finite-dimensional C*-algebra is isomorphic to a (finite) direct sum of matrix algebras. Extending this, one can consider infinite-dimensional C*-algebras that can be approximated by finite-dimensional ones. These are the so-called AF ("almost finite-dimensional") C*-algebras, and they were classified by George Elliott in the 1970's using K-theory [34]. Starting from there, Elliott initiated an ambitious program to classify ever larger classes of C*-algebras by K-theoretic data. Only recently this so-called *classification program* has culminated in the classification theorem for a large class of C*-algebras, see e.g. [40].

Yet the classification program is inherently limited to C*-algebras that are in some ways "well-behaved". A prominent condition in the classification theorem is *nuclearity.* A C*-algebra is nuclear if it can be approximated in a particular way by finite-dimensional C*-algebras. Many natural examples of C*-algebras are nuclear, for instance all commutative C*-algebras, all finite-dimensional C*-algebras and all AF C*-algebras. However, free products of C*-algebras tend to be non-nuclear. As an example, the unital free product $\mathbb{C}(S^1) *_{\mathbb{C}} \mathbb{C}(S^1)$ of two copies of the continuous functions on the unit circle S^1 is not-nuclear, which is an echo of the famous Banach–Tarski paradox. In the first part of this thesis we solve the classification problem for a particular class of C*-algebras that are generally not nuclear and thus fall outside the realm of Elliott's classification program.

1.1.1. Two Partitions of Unity

The C*-algebras that we investigate are generated by two partitions of unity subject to particular orthogonality relations prescribed by a bipartite graph. Given a bipartite graph $G = (U, V, E)$ with disjoint (finite) vertex sets U and V and edge set $E \subset \{\{u, v\} \mid u \in U, v \in V\}$, we consider the universal C*-algebra $C^*(G)$ that is generated by projections p_x with $x \in U \cup V$ satisfying the two relations

$$\sum_{u \in U} p_u = 1 = \sum_{v \in V} p_v, \tag{GP1}$$

$$p_u p_v = 0 \ \text{ if } \{u, v\} \notin E. \tag{GP2}$$

Thus, the families $\{p_u\}_{u \in U}$ and $\{p_v\}_{v \in V}$ consist of pairwise orthogonal projections which add up to the unit of $C^*(G)$, and this is what we call a partition of unity. We call $C^*(G)$ the *bipartite graph C^*-algebra* associated to G.

These C*-algebra generalize the universal (unital) C*-algebra generated by two projections. Famously, Halmos' described the general structure of two projections in a Hilbert space in his Two Projection Theorem [45], and Pedersen provided an explicit description of the universal C*-algebra of two projections as algebra of particular continuous functions from $[0, 1]$ into the 2×2-matrices [67]. In Theorem 4.4.3 and Theorem 7.6.4 we generalize both results to bipartite graph C*-algebras.

Our interest in these algebras stems from their connection to the recently introduced hypergraph C^*-algebras which have first been investigated by Trieb, Weber and Zenner in [85]. Hypergraph C*-algebras are a novel generalization of graph C*-algebras and not much is known about them. Open questions include their classification or which hypergraph C*-algebras are nuclear. The latter question was studied by Moritz Weber and the author in [80]. There, a partial result was obtained: In order to tell which hypergraph C*-algebras are nuclear it suffices to know which *undirected* hypergraph C^*-algebras are nuclear. In fact, the latter are exactly the algebras we study in the present thesis (see Proposition 5.3.1), though we prefer the language of bipartite graphs for the sake of clarity.

1.1.2. Contributions

Our primary goal is to solve the classification problem for bipartite graph C*-algebras. Before we do this, we collect some results which offer different perspectives on these algebras. First, we provide an alternative set of generators for the bipartite graph C*-algebra $C^*(G)$. By definition, $C^*(G)$ is generated by projections associated to the vertices of the graph G, but we show that it is as well generated by contractions associated to the edges of G - satisfying particular relations, of course. In principle, one can extend this observation and find generators associated to paths in G of any fixed length k. This would, however, increase both the number of generators and relations unfavorably.

Proposition A (Proposition 4.3.1)**.** Let $G = (U, V, E)$ be a bipartite graph. Then $C^*(G)$ is the universal C^*-algebra generated by a family of elements $(x_e)_{e\in E}$ satisfying

$$x_e^* x_f = 0 \qquad \text{if } e \cap f \cap U = \emptyset, \tag{GC1}$$
$$x_e x_f^* = 0 \qquad \text{if } e \cap f \cap V = \emptyset, \tag{GC2}$$
$$\left(\sum_{e\in E} x_e^*\right) x_f = x_f, \tag{GC3}$$
$$x_e \left(\sum_{f\in E} x_f^*\right) = x_e, \tag{GC4}$$

for all edges $e, f \in E$, respectively. In particular, the x_e are contractions which satisfy $x_e x_e^* x_e = x_e^2$.

Next, we investigate what we can say about concrete projections $(P_x)_{x\in U\cup V} \subset B(\mathcal{H})$ on some Hilbert space $\mathcal{H}$ that satisfy the relations (GP1) and (GP2). We call such projections a G-projection family. Following earlier work by Vasilevski [87] we obtain the following generalization of Halmos' Two Projection Theorem to G-projection families.

Theorem B (Theorem 4.4.3)**.** Every G-projection family $(P_x)_{x\in U\cup V}$ in generic position on some Hilbert space $\mathcal{H}$ associated to a connected bipartite graph G is (up to unitary equivalence) of the form

$$P_u = (C_{uv_1}^* C_{uv_2})_{v_1,v_2\in V} \in M_V(\mathcal{B}(\mathcal{K})),$$
$$P_v = (\delta_{v_1 v}\delta_{v_2 v})_{v_1,v_2\in V} \in M_V(\mathcal{B}(\mathcal{K})),$$

for operators $(C_{uv})_{u\in U, v\in V}$ on a Hilbert space $\mathcal{K}$ that satisfy particular relations including $C_{uv} = 0$ if $\{u, v\} \notin E$.

The conditions imposed on the operators C_{uv} can be found in the statement of Theorem 4.4.3. They generalize the condition $C^2 + S^2 = I$ from Halmos' Theorem and incorporate the graph structure by asking $C_{uv} = 0$ whenever $\{u, v\} \notin E$.

Finally, we solve the classification problem and answer the following question: When is $C^*(G) \cong C^*(G')$ for two bipartite graphs G and G'. For that we consider the subspace $\mathrm{Spec}_{\leq 2}(C^*(G))$ of the spectrum of $C^*(G)$ which contains only the (equivalence classes of) one- and two-dimensional irreducible representations. The structure of this space can be easily read off from the bipartite graph G. In fact, the one-dimensional irreducible representations correspond to edges of the bipartite graph, while the two-dimensional irreducible representations correspond to subgraphs that are isomorphic to $K_{2,2}$, see Lemma 6.1.3 and Lemma 6.1.6. Then we obtain the following theorem.

Theorem C (Theorem 6.2.6)**.** We have

$$C^*(G) \cong C^*(G') \quad \Leftrightarrow \quad \mathrm{Spec}_{\leq 2}(C^*(G)) \cong \mathrm{Spec}_{\leq 2}(C^*(G')).$$

Recall that bipartite graph C*-algebras are special cases of hypergraph C*-algebras. Hence, Theorem C provides a first step towards a classification of hypergraph C*-algebras.

As mentioned before, another open question about hypergraph C*-algebras is which of them are nuclear. For bipartite graph C*-algebras it is an easy observation that $C^*(G)$ is not nuclear as soon as $K_{2,3} \subset G$, where $K_{2,3}$ is the complete bipartite graph with 2 and 3 vertices. It is unclear if the converse is true, i.e. whether $C^*(G)$ is nuclear if $K_{2,3} \not\subset G$. Towards answering this question, we investigate a particular class of bipartite graphs G with $K_{2,3} \not\subset G$: hypercube graphs.

For the hypercubes Q_n we are able to provide an explicit description of the associated bipartite graph C*-algebras $C^*(Q_n)$. If $n = 2$, then $C^*(Q_2)$ is the universal C*-algebra of two projections. By a theorem of Pedersen [67] this C*-algebra is given by

$$C^*(p, q) \cong \{f \in C([0, 1], M_2) \mid f(0), f(1) \text{ are diagonal}\}.$$

To generalize this, we replace the unit interval $[0, 1]$ by the standard simplex $\Delta_{n-1} = \{\mathbf{t} = (t_0, \ldots, t_{n-1}) \in [0, 1]^n \mid \sum_{i=0}^{n-1} t_i = 1\}$ and the matrix algebra M_2 by the matrix algebra $M_{2^{n-1}}$. In Pedersen's result, functions $f \in C^*(p, q) \subset C([0, 1], M_2)$ had to satisfy the condition that $f(0)$ and $f(1)$ are diagonal matrices. We replace that condition with a more complex one which we summarize by saying that $f(\mathbf{t})$ must be in $\mathbf{t}$-block diagonal form for all $\mathbf{t} \in \Delta_{n-1}$. The precise definition of $\mathbf{t}$-block diagonal form can be found in Notation 7.6.1.

Theorem D (Theorem 7.6.4)**.** The C*-algebra $C^*(Q_n)$ is isomorphic to

$$A_n := \{f \in C(\Delta_{n-1}, M_{2^{n-1}}) \mid f(\mathbf{t}) \text{ is in } \mathbf{t}\text{-block diagonal form for all } \mathbf{t} \in \Delta_{n-1}\}.$$

This theorem entails that the algebras $C^*(Q_n)$ are nuclear, supporting the hypothesis that $K_{2,3} \not\subset G$ implies nuclearity of $C^*(G)$.

At the end of the first part of this thesis, we apply our findings on the hypercube algebras $C^*(Q_n)$ to an open problem about magic isometries. A magic isometry is a matrix $P = (P_{ij}) \in M_{m,n}(A)$ with entries from a unital C*-algebra A such that every P_{ij} is a projection and $PP^* = I_m$. If $m = n$, the matrix P is also called a magic unitary. In [7], Banica, Skalski and Sołtan asked if any $m \times n$ magic isometry with $m < n$ can be completed to a $n \times n$ magic unitary by adding $(n - m)$ rows. In the special case, where $m = 2$ and $n = 4$, it turns out that the universal C*-algebra generated by the entries of a 2×4 magic isometry is isomorphic to the hypercube algebra $C^*(Q_3)$. Using our description of $C^*(Q_3)$, we answer the question of Banica, Skalski and Sołtan in the positive.

Proposition E (Proposition 7.7.7)**.** Every 2×4-magic isometry can be completed to a 4×4-magic unitary matrix.

1.2. Part II: Derived quantum graphs

By a famous result of Erdős and Rényi, almost all finite graphs have no symmetries [35]. Yet most graphs that one encounters in real life exhibit some symmetry, starting from the cyclic graphs to more complex ones like the Petersen graph. How can one exploit these symmetries? In the 1970s, Gross and Tucker developed a method to describe graphs with symmetries in terms of smaller labeled graphs in an insightful way [43, 42]. The main objects of their theory are so-called *voltage graphs*. A voltage graph is a pair $(\tilde{\Gamma}, \lambda)$ consisting of a finite (not necessarily simple) graph $\tilde{\Gamma}$ together with a labeling $\lambda : E_{\tilde{\Gamma}} \to G$ of its edges by elements of a finite group G. Then, they introduced the *derived graph* $\tilde{\Gamma}^\lambda$ as a particular graph on the vertex set $V_{\tilde{\Gamma}} \times G$ where the edges are determined by the labeling λ. Gross and Tucker's main theorem states that a graph Γ is isomorphic to a derived graph $\tilde{\Gamma}^\lambda$ if and only if there is a free action of the group G on Γ.

Voltage and derived graphs have found applications in a number of settings, e.g. for generating graph coverings [44], for constructing large graphs with small degree and diameter [14] or for lifting graph automorphisms [59]. Kumjian and Pask used voltage graphs to decompose graph C*-algebras into crossed products up to stable isomorphism [54].

In the second part of this thesis we generalize voltage graphs, derived graphs and the Gross–Tucker theorem to *quantum graphs*. Quantum graphs are a recent generalization of classical graphs where the (finite) vertex set is replaced with an arbitrary finite-dimensional C*-algebra. In the spirit of Gelfand duality a finite-dimensional C*-algebra is viewed as a noncommutative or quantum version of a classical finite set. For a number of reasons, though, a quantum set is indeed a pair (B, ψ) of a finite-dimensional C*-algebra and a particular functional ψ.

Quantum graphs originate in quantum information theory where they were introduced by Duan, Severini and Winter to generalize confusability graphs from classical communication channels to quantum channels [30]. From a purely mathematical perspective Kuperberg and Weaver discovered quantum graphs as particular cases of quantum relations [56, 97, 96]. A completely different perspective on quantum graphs was later proposed by Musto, Reutter and Verdon who described quantum graphs by so-called quantum adjacency matrices [64]. Let us also mention work of Kornell which shows that quantum graphs appear naturally in a quantum version of first-order logic [52, 53]. Most importantly, all approaches to quantum graphs result in essentially the same object. When the underlying quantum set is tracial, meaning that the functional ψ is a trace, this had been proved by Weaver and Musto–Reutter–Verdon. For the non-tracial case different equivalences have been laid out by Daws and Wasilewski [27, 93].

What are examples of quantum graphs? For a start, every classical graph on n vertices is a quantum graph on the quantum set $(\mathbb{C}^n, \psi_n)$ for a suitable functional ψ_n, see Examples 8.1.4 and 8.2.4. Undirected quantum graphs on the quantum set $(M_2, 2\mathrm{Tr})$ were classified by Gromada and Matsuda, and it turned out that, up to isomorphism, there exist exactly four such quantum graphs which are quantum isomorphic to classical graphs [41, 62]. In this thesis we extend their classification to loopfree *directed* quantum graphs on $(M_2, 2\mathrm{Tr})$. This is just a special case of a more extensive classification of the author and Nina Kiefer [50]. The obtained result is the following theorem. (For a more detailed statement we refer to Theorem 9.3.5.)

Theorem F (Theorem 9.3.5)**.** Every loopfree directed quantum graph on $(M_2, 2\mathrm{Tr})$ is isomorphic to exactly one of the quantum graphs given by the quantum adjacency matrices

$$
\begin{aligned}
A_0 &= 0,\\
A_1^{(\beta)} &= \frac{1}{1+\beta^2}\begin{pmatrix} 0 & 0 & 0 & \beta_+^2 \\ 0 & 0 & \beta_+\beta_- & 0 \\ 0 & \beta_+\beta_- & 0 & 0 \\ \beta_-^2 & 0 & 0 & 0 \end{pmatrix},\\
A_2^{(\beta)} &= \frac{1}{1+\beta^2}\begin{pmatrix} 0 & 0 & 0 & \beta_-^2 \\ 0 & 0 & -\beta_+\beta_- & 0 \\ 0 & -\beta_+\beta_- & 0 & 0 \\ \beta_+^2 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\\
A_3 &= \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix},
\end{aligned}
$$

where $\beta_+ := 1+\beta$ and $\beta_- := 1-\beta$ for $\beta \in [0,1]$.

Quantum graphs have been applied to a range of different topics. In quantum information theory one is most interested in computing values such as independence or chromatic numbers of quantum graphs, see e.g. [83, 51, 17, 38, 11]. Nonlocal games, a central tool in quantum information theory, for quantum graphs were studied in [17, 21, 20, 39, 31]. Further, quantum graphs have important connections to the theory of quantum groups: Wasilewski introduced quantum Cayley graphs [93], and it was shown that every finite quantum group is the quantum automorphism group of a quantum graph [18]. Other applications of quantum graphs to quantum groups include [36, 15, 22]. Yet in another area, quantum graphs can be used to construct C*-algebras which generalize the well-known Cuntz-Krieger and graph C*-algebras [16, 19, 46].

This thesis introduces the notion of *voltage quantum graphs* and their *derived quantum graphs* (Definition 12.1.1). Intuitively, a voltage quantum graph is a quantum graph with an edge labeling by elements of a finite abelian group G. In precise terms, it is a family $\tilde{A} = (\tilde{A}_g)_{g\in G}$ of quantum adjacency matrices on a quantum set $(\tilde{B}, \tilde{\psi})$. One should think of every $\tilde{A}_g$ as the quantum graph of all edges that are labeled by g. Classically, the vertex set of the derived graph is the Cartesian product of the vertex set $\tilde{V}$ of the voltage graph and the group G. In the quantum setting we have more flexibility: Given any action $\hat{\alpha}$ of the dual group $\hat{G}$ on the voltage quantum graph $\tilde{A}$, we obtain a derived quantum graph

$$A := \tilde{A} \rtimes_{\hat{\alpha}} \hat{G},$$

which is defined on the *crossed product quantum set* (Definition 11.2.1)

$$(\tilde{B}, \tilde{\psi}) \rtimes_{\hat{\alpha}} \hat{G} := (\tilde{B} \rtimes_{\hat{\alpha}} \hat{G}, \psi_{\rtimes}).$$

Thus, we also call $\tilde{A}$ a $(G, \hat{\alpha})$-voltage quantum graph. One family $\tilde{A} = (\tilde{A}_g)_{g\in G}$ is a $(G, \hat{\alpha})$-voltage quantum graph for every action $\hat{\alpha}$ of $\hat{G}$ on the quantum set $(\tilde{B}, \tilde{\psi})$ which preserves all adjacency matrices $\tilde{A}_g$[1]. The classical voltage and derived graphs of Gross and Tucker are recovered as (G, triv)-voltage quantum graphs on $(\mathbb{C}^n, \psi_n)$ where triv is the trivial action of $\hat{G}$ on $(\mathbb{C}^n, \psi_n)$ (see Proposition 12.3.1).

Every classical voltage graph is in particular a voltage quantum graph. It is a remarkable feature of our construction that one can start with a classical voltage graph $\tilde{A}$ and obtain a true quantum graph, i.e. one that is not a classical graph. For instance, let $\tilde{A}$ be a classical voltage graph on two vertices. If $\tilde{A}$ is preserved under the action $\hat{\alpha}$ of $\hat{\mathbb{Z}}_2$ which swaps the two vertices, then $\tilde{A}$ is also a $(\mathbb{Z}_2, \hat{\alpha})$-voltage quantum graph. The crossed product quantum set with respect to the action $\hat{\alpha}$ is

$$(\mathbb{C}^2, \psi_2) \rtimes_{\hat{\alpha}} \hat{\mathbb{Z}}_2 \cong (M_2, 2\mathrm{Tr}).$$

Accordingly, the derived quantum graph $\tilde{A} \rtimes_{\hat{\alpha}} \hat{\mathbb{Z}}_2$ is a quantum graph on the quantum set $(M_2, 2\mathrm{Tr})$. In fact, as mentioned before, there are only four non-isomorphic (undirected, loopfree) quantum graphs on $(M_2, 2\mathrm{Tr})$, and we show that they are all obtained in this way (Proposition 12.5.1).

Further, Matsuda proved that all four loopfree undirected quantum graphs on $(M_2, 2\mathrm{Tr})$ are quantum isomorphic to classical graphs [62]. We recover this quantum isomorphism as a special case of a much more general phenomenon.

[1] Indeed, an action of $\hat{G}$ on a quantum (voltage) graph is nothing else but an action on the underlying quantum set that preserves the additional quantum graph structure.

Theorem G (Theorem 12.4.1)**.** Assume that $\tilde{A}$ is a $(G, \hat{\alpha})$-voltage quantum graph on $(\tilde{B}, \tilde{\psi})$. Then $\tilde{A}$ is also a (G, triv)-voltage quantum graph, and we have a quantum isomorphism

$$\tilde{A} \rtimes_{\hat{\alpha}} \hat{G} \cong_q \tilde{A} \rtimes_{\mathrm{triv}} \hat{G},$$

where triv is the trivial action of $\hat{G}$ on $(\tilde{B}, \tilde{\psi})$.

Finally, the main result of this work is a generalization of the Gross–Tucker theorem to quantum graphs: Classically, a graph Γ is isomorphic to a derived graph $\tilde{\Gamma}^\lambda$ if and only if there is a free action of G on Γ such that $\tilde{\Gamma} \cong \Gamma/G$, i.e. the (unlabeled) voltage graph is isomorphic to the quotient graph. The first problem in generalizing this theorem is to find an analogue of a free action on a quantum set. Here, the answer is given by a theorem of Landstad [58] which precisely characterizes when a given C*-algebra B is isomorphic to a crossed product algebra $\tilde{B} \rtimes_{\hat{\alpha}} \hat{G}$ (see Theorem 11.4.1). Note that the conditions in Landstad's theorem are stronger than mere freeness of the action which has been generalized to the setting of C*-algebras in various ways [69, 74, 9].

Theorem H (Theorem 13.0.2)**.** Let $A : B \to B$ be a quantum graph on the quantum set (B, ψ) and let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite abelian group G on A which has the following properties:

1. There is a representation $(u_\chi)_{\chi \in \hat{G}} \subset B$ of the dual group $\hat{G}$ such that
$$\alpha_g(u_\chi) = \chi(g) u_\chi$$
holds for all $g \in G$ and $\chi \in \hat{G}$.
2. One has
$$\psi \mathrm{Ad}(u_\chi) = \psi \quad \text{as well as} \quad \mathrm{Ad}(u_\chi) A = A \mathrm{Ad}(u_\chi)$$
for all $\chi \in \hat{G}$.

Then there is a voltage quantum graph $\tilde{A}$ on $(B^\alpha, \tilde{\psi})$ such that

$$A \cong \tilde{A} \rtimes_{\hat{\alpha}} \hat{G},$$

where $B^\alpha \subset B$ is the fixed-point subalgebra, $\hat{\alpha} : \hat{G} \to \mathrm{Aut}(B^\alpha, \tilde{\psi})$ is given by $\hat{\alpha}_\chi(b) = u_\chi b u_\chi^*$ for all $b \in B^\alpha$ and $\chi \in \hat{G}$, and $\tilde{\psi} = \frac{1}{|G|} \psi|_{B^\alpha}$.

1.3. Included publications

The first part of this thesis consists mainly of material from two articles of the author [78, 77] which are to be published in peer-reviewed journals. During his PhD, the author co-supervised the Master's thesis of Fabian Selzer [82] where a similar theorem to Theorem 7.6.4 was obtained by different methods.

The second part of this thesis is based on the preprint [79] which is co-authored by the author and Mariusz Tobolski. Further, it contains material from the publication [50] which is a joint work of the author and Nina Kiefer. We explain the contributions to these publications for every chapter:

- Chapter 8 collects background material on quantum graphs and contains no original work.
- Chapter 9 is based on [50] which is a joint work of the author and Nina Kiefer.
- Chapter 10 recalls the well-known theory of Gross and Tucker and contains no original work.
- Chapter 11 describes crossed product quantum sets and a version of Landstad's duality theorem for quantum sets. The construction of the quantum set functional on the crossed product C*-algebra is due to the author. The idea to use Landstad's theorem for quantum sets is due to Mariusz Tobolski, while the proof of the version of Landstad's theorem for quantum sets is the author's work. The quantum isomorphism between different crossed product quantum sets is the author's original work.
- Chapter 12 introduces voltage quantum graphs and derived quantum graphs. The definition of voltage quantum graphs and derived quantum graphs is the author's work, as well as the quantum isomorphism between different derived quantum graphs. The example of a derived quantum graph in Example 12.3.4 traces back to a discussion between the author and Mariusz Tobolski at University of Wrocław in 2025. Similarly, the idea to obtain quantum graphs on $(M_2, 2\mathrm{Tr})$ as derived quantum graphs stems from a discussion between the author and Mariusz Tobolski, while the proof in Section 12.5 is the author's work.
- Chapter 13 contains the main result of the second part of this thesis, a generalization of the Gross–Tucker theorem to quantum graphs. The formulation of the theorem and its proof are the author's original work.

1.4. Notation

Let us introduce some general notation that is used throughout this thesis. More specialized notation is introduced when it is appropriate. A list of symbols can be found at the end of the thesis.

The symbols $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ denote the sets of natural, integer, real and complex numbers, respectively. We reserve the symbol X for topological spaces, the symbols B, B_1, B_2 for C*-algebras and the symbol $\mathcal{H}$ for Hilbert spaces. The opposite C*-algebra of B is denoted by B^{op}, this is the C*-algebra obtained from B by changing the multiplication to $b \cdot_{B^{\mathrm{op}}} c := c \cdot_B b$. The algebra of complex $n \times n$ matrices is denoted by M_n, and $B(\mathcal{H})$ denotes the bounded operators on $\mathcal{H}$. The identity operator on a C*-algebra B

is denoted by Id_B and the identity matrix in M_n by I_n or simply I. The unit in a C*-algebra is denoted by 1_B or simply 1. The symbols E_{ij} denote the standard matrix units in M_n, while we denote by e_i the standard basis vectors in $\mathbb{C}^n$. We also write e_v or $e_{v,g}$ for the natural indicator functions in $C(V)$ and $C(V \times G)$, respectively. We denote by Tr the un-normalized trace on M_n and by tr the normalized trace, i.e. $\mathrm{tr} = \frac{1}{n}\mathrm{Tr}$. The symbols φ, ψ are used for $*$-homomorphisms between C*-algebras, but in the second part ψ is reserved for a (positive faithful) functional on a C*-algebra. The symbol $\otimes$ denotes the tensor product of C*-algebras, and we generally are in a situation where minimal and maximal tensor product coincide. At some places $\otimes$ denotes the Kronecker product of matrices, and it will be clear from the context which one is meant.

In the first part of the thesis, projections on Hilbert spaces are typically denoted by P, Q and projections in a C*-algebra by p, q. A universal C*-algebra is denoted by $C^*(x_i \mid R_j)$ where x_i are the generators and R_j are the relations. Further, $G = (U, V, E)$ denotes a finite bipartite graph with adjacency relation $u \sim v$, neighborhood sets $\mathcal{N}(x)$ and complete bipartite graphs $K_{m,n}$. Paths in bipartite graphs are denoted by μ, ν with source and range $s(\mu), r(\mu)$, respectively. Hypergraphs are denoted by $\mathrm{H}\Gamma = (E^0, E^1, r, s)$, and the associated hypergraph C*-algebras by $C^*(\mathrm{H}\Gamma)$. The n-dimensional hypercube graph is denoted by $Q_n = (U_n, V_n, E_n)$, its vertices are binary numbers denoted by i, j. By Δ_n we denote the n-dimensional standard simplex with elements $\mathbf{t} = [t_0, \ldots, t_{n-1}]$.

In the second part of the thesis, G denotes a finite abelian group with dual group $\hat{G}$. The finite cyclic groups are denoted by $\mathbb{Z}_n$. Elements of G are denoted by $h, g, \ldots$, while elements of $\hat{G}$ are denoted by $\chi, \xi, \ldots$; the unit of the group G is generally denoted by 1, but the unit in $\mathbb{Z}_n$ is, naturally, 0. For $\chi, \xi \in \hat{G}$, $\delta_{\chi=\xi}$ denotes the Kronecker delta and similarly for subscripts h, g.

We use the symbols α and $\hat{\alpha}$ to denote actions of G and $\hat{G}$, respectively, on C*-algebras. The fixed point algebra of the action α is denoted by B^α, and the crossed product of a C*-algebra B by the action α is denoted by $B \rtimes_\alpha G$. We are generally in a situation where the reduced crossed product is equal to the full crossed product. If $u \in B$ is unitary, then we denote by $\mathrm{Ad}(u)$ its action on B by conjugation. A quantum set is generally written as a pair (B, ψ) with automorphism set $\mathrm{Aut}(B, \psi)$, and crossed products $(B, \psi) \rtimes_\alpha G$. Quantum graphs are denoted by their adjacency matrix $A : B \to B$ and their automorphism group is denoted by $\mathrm{Aut}(A)$. The symbol $\cong$ denotes isomorphisms of C*-algebras, quantum sets or quantum graphs, while $\cong_q$ denotes quantum isomorphisms of quantum sets or quantum graphs.

1.5. Outline

In the next **Chapter 2** we lay out some background on the general theory of C*-algebras. Then, we start the first part of the thesis by recalling in **Chapter 3** a number of classical results from the theory of two projections. In **Chapter 4** we introduce bipartite graph C*-algebras together with some first properties. **Chapter 5** is about hypergraph C*-algebras and explains their connection to bipartite graph C*-algebras. Then we present in **Chapter 6** a first main result: the classification of bipartite graph C*-algebras in terms of their spectrum. The next **Chapter 7** is devoted to hypercube C*-algebras, the bipartite graph C*-algebras associated to the hypercube graphs.

The second part of this thesis is on derived quantum graphs. In **Chapter 8** we lay out the basic definitions from the theory of quantum graphs. Then, **Chapter 9** recalls and extends classification results for quantum graphs on the matrix algebra M_2. Gross and Tucker's classical construction of derived graphs is explained in **Chapter 10**. In order too generalize this to quantum graphs, we first describe crossed product quantum sets in **Chapter 11**, and then go on to introduce voltage and derived quantum graphs in **Chapter 12**. We end the thesis with a main result generalizing the classical theorem of Gross and Tucker in **Chapter 13**.

2. Preliminaries on C*-algebras

In this chapter, we collect basic definitions and constructions about C*-algebras. Our reference is generally Blackadar's monograph [10] but we will also refer to Dixmier's book [28] later on.

2.1. Definition and structure theorems

Definition 2.1.1. A C*-algebra is a complex associative algebra B together with a map $*: B \to B$ (called "involution") satisfying

$$(b^*)^* = b, \quad (\lambda b + \mu c)^* = \overline{\lambda} b^* + \overline{\mu} c^*, \quad (bc)^* = c^* b^*, \qquad \text{for all } b, c \in B, \lambda, \mu \in \mathbb{C},$$

and a norm $\|\cdot\|$ such that $(B, \|\cdot\|)$ is a complete normed space and the C*-identity

$$\|bb^*\| = \|b\|^2$$

holds for all $b \in B$. The C*-algebra B is called

a) *finite-dimensional* if it is finite-dimensional as a complex vector space,

b) *unital* if it contains a unit 1_B such that $1_B b = b 1_B = b$ holds for all $b \in B$,

c) *commutative* if $bc = cb$ holds for all $b, c \in B$.

Let us collect some examples of C*-algebras.

Example 2.1.2.

1. If X is a locally compact space, then $C_0(X)$, the algebra of continuous complex-valued functions on X vanishing at infinity, is a (commutative) C*-algebra with pointwise operations and the supremum norm, i.e.

   $$(fg)(x) = f(x)g(x), \qquad (f^*)(x) = \overline{f(x)}, \qquad \|f\| = \sup_{x \in X} |f(x)|.$$

   Evidently, $C_0(X)$ is unital if and only if X is a compact space.

2. If $\mathcal{H}$ is a Hilbert space, then the bounded operators on $\mathcal{H}$ together with composition as multiplication, the operator norm and the adjoint operation form a (unital, generally non-commutative) C*-algebra $B(\mathcal{H})$. In particular, the matrix algebras M_n are C*-algebras.

3. Further, every subalgebra of $B(\mathcal{H})$ that is closed with respect to the operator norm and the adjoint operation is a C*-algebra as well.

Famously, these examples cover all possible C*-algebras thanks to two fundamental structure theorems.

Theorem 2.1.3 (Gelfand–Naimark)**.** *Every commutative C*-algebra is isomorphic to $C_0(X)$ for some locally compact Hausdorff space X.*

Theorem 2.1.4 (Gelfand–Naimark–Segal)**.** *Every C*-algebra A is isomorphic to a closed sub-C*-algebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

The latter theorem is proved using the GNS-construction which we will describe in detail in a later section.

2.2. Basic notions

In this section, we collect some basic notions about C*-algebras.

Definition 2.2.1. Let B be a C*-algebra. An element $b \in B$ is called

a) *self-adjoint* if $b = b^*$,

b) *positive*, written $b \geq 0$, if $b = y^*y$ for some $y \in B$,

c) a *projection* if $b = b^* = b^2$,

d) a *contraction* if $\|b\| \leq 1$,

e) an *isometry* if B is unital and $b^*b = 1_B$,

f) a *unitary* if B is unital and $bb^* = b^*b = 1_B$.

A finite family of projections $\{p_i \mid i \in I\} \subset B$ is called a *partition of unity* if $\sum_{i\in I} p_i = 1_B$. Two projections $p, q \in B$ are called *orthogonal* if $pq = 0$. The positive elements of B induce a partial order on B via $b_1 \leq b_2 :\Leftrightarrow b_2 - b_1 \geq 0$ for $b_1, b_2 \in B$.

Definition 2.2.2. Let B_1, B_2 be C*-algebras. A linear map $\varphi : B_1 \to B_2$ is called

a) *faithful* if $\varphi(b^*b) = 0$ implies $b = 0$ for all $b \in B_1$,

b) *unital* if B_1 and B_2 are unital C*-algebras and $\varphi(1_{B_1}) = 1_{B_2}$,

c) *positive* if $\varphi(b) \geq 0$ holds for all positive $b \in B_1$,

d) a *functional* if $B_2 = \mathbb{C}$,

e) a *state* if it is a positive functional with

$$\|\varphi\| := \sup_{\|b\|=1} |\varphi(b)| = 1.$$

Further, φ is called a $*$-*homomorphism* if it satisfies

$$\varphi(bc) = \varphi(b)\varphi(c), \qquad\qquad \varphi(b^*) = \varphi(b)^*,$$

for all $b, c \in B_1$, i.e. if it is multiplicative and symmetric. A bijective $*$-homomorphism is called a $*$-*isomorphism*. A $*$-homomorphism into the C*-algebra of bounded operators $B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is called a *representation*.

The $*$-homomorphisms are the natural maps between C*-algebras. They interact in the expected way with the following notions/constructions.

Definition 2.2.3. Let B_1 and B_2 be C*-algebras. Then, we say that

a) B_2 is isomorphic to B_1, written $B_2 \cong B_1$, if there is a $*$-isomorphism $\varphi : B_1 \to B_2$,

b) B_2 is a subalgebra of B_1, written $B_2 \subset B_1$, if B_2 is a linear subspace of B_1 that is closed under multiplication and involution,

c) B_2 is an ideal in B_1 if it is a subalgebra that is closed under left and right multiplication with arbitrary elements of B_1,

d) B_2 is a quotient of B_1 if there is an ideal $I \subset B_1$ such that elements of B_2 are equivalence classes $[b]$ of B_1 under the equivalence relation

$$b \sim b' \quad :\Leftrightarrow \quad b - b' \in I,$$

and one has $\lambda[b] + \mu[c] = [\lambda b + \mu c]$, $[b][c] = [bc]$, $[b]^* = [b^*]$ as well as

$$\|[b]\| = \inf_{x \in I} \|b + x\|,$$

Further,

a) a tensor product $B_1 \otimes B_2$ is given by equipping the algebraic tensor product $B_1 \odot B_2$, i.e. the tensor product of B_1 and B_2 as algebras, with a C*-norm and completing it with respect to this norm,

b) the direct sum $B_1 \oplus B_2$ is given by the set of pairs (b_1, b_2) with $b_1 \in B_1$ and $b_2 \in B_2$ together with pointwise operations and the norm $\|(b_1, b_2)\| := \max\{\|b_1\|, \|b_2\|\}$.

Generally, two C*-algebras B_1 and B_2 admit various non-isomorphic tensor products. More well-behaved C*-algebras are called nuclear.

Definition 2.2.4. A C*-algebra B_1 is called *nuclear* if for every other C*-algebra B_2 there is a unique C*-norm on the algebraic tensor product $B_1 \odot B_2$.

Example 2.2.5. All finite-dimensional and all commutative C*-algebras are nuclear.

2.3. GNS-construction

An early problem in the theory of C*-algebras was to reconcile two different definitions: First, an *abstract* C*-algebra is a complex associative algebra with involution and a complete norm satisfying the C*-identity. Second, a *concrete* C*-algebra is a norm-closed subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. The GNS-construction, named after Gelfand, Naimark and Segal, provides a way to represent abstract C*-algebras concretely. Recall that a representation of a C*-algebra is a $*$-homomorphism from B into the space of bounded operators $B(\mathcal{H})$ on some Hilbert space $\mathcal{H}$. The GNS-construction provides a particular faithful representation for a given abstract C*-algebra.

Definition 2.3.1 (GNS-construction)**.** Let B be a C*-algebra with a positive functional $\psi : B \to \mathbb{C}$. Then

$$\langle b, c\rangle_\psi := \psi(b^* c) \quad \text{for } b, c \in B,$$

is a pre-inner product on B. It induces naturally an inner product on the quotient $B/\{b \in B \mid \psi(b^*b) = 0\}$ whose elements we denote by $[b]$ for $b \in B$. The completion of this space is a Hilbert space $\mathcal{H}_\psi = L^2(B, \psi)$. For every $b \in B$ one can extend the map

$$\mathcal{H}_\psi \ni [c] \mapsto [bc] \in \mathcal{H}_\psi \quad \text{for } c \in B,$$

to a bounded operator $\pi_\psi(b) \in B(\mathcal{H}_\psi)$. The map

$$\pi_\psi : B \to B(\mathcal{H}_\psi), b \mapsto \pi_\psi(b)$$

is a $*$-homomorphism called the *GNS-representation* associated to ψ.

Remark 2.3.2. If B is finite-dimensional and ψ is faithful, then $L^2(B, \psi)$ is nothing else but the linear space B equipped with the inner product $\langle b, c\rangle_\psi = \psi(b^* c)$.

2.4. Spectrum of a C*-algebra

By the Gelfand–Naimark Theorem, any commutative C*-algebra B is of the form $B \cong C_0(X)$ for some locally compact Hausdorff space X. In this case, the space X is given by the characters of the C*-algebra, i.e. by $*$-homomorphisms from B into $\mathbb{C}$. For noncommutative C*-algebras, characters are generalized to irreducible representations and the space X is generalized to the spectrum $\mathrm{Spec}(B)$ of the C*-algebra.

Definition 2.4.1. Let B be a C*-algebra. A *representation* $\rho : B \to B(\mathcal{H})$ of B is called *irreducible* if there is no non-trivial closed subspace $\mathcal{K} \subset \mathcal{H}$ with $\rho(B)\mathcal{K} \subset \mathcal{K}$. Two irreducible representations ρ_1, ρ_2 are unitarily equivalent if there is a unitary operator U between the respective Hilbert space such that $\rho_1(b) = U^*\rho_2(b)U$ holds for all $b \in B$. Two representations ρ_1 and ρ_2 on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, give rise to a naturally defined representation $\rho_1 \oplus \rho_2$ on $\mathcal{H}_1 \oplus \mathcal{H}_2$.

We say that a representation $\rho : B \to B(\mathcal{H})$ is k-dimensional for $k \in \mathbb{N}$ if $\dim(\mathcal{H}) = k$. A C*-algebra B where all irreducible representations have a uniformly bounded dimension have a special name.

Definition 2.4.2. A C*-algebra B is called (n-)subhomogeneous if every irreducible representation $\rho : B \to B(\mathcal{H})$ has dimension less or equal than n with $n \in \mathbb{N}$.

We will use the following proposition later to conclude that all hypercube C*-algebras are nuclear. For a proof see [10].

Proposition 2.4.3. *Every subhomogeneous C*-algebra is nuclear.*

The set of all irreducible representations of a C*-algebra B can be equipped with a suitable topology, and the obtained space is called the spectrum of B.

Definition 2.4.4. Let B be a C*-algebra.

a) The primitive ideal space $\mathrm{Prim}(B)$ is the set of all kernels $\rho^{-1}(0)$ of irreducible representations ρ of B, equipped with the hull-kernel topology, i.e. the closed sets are given by

$$\overline{S} := \{J \in \mathrm{Prim}(B) \mid J \supset \bigcap_{I \in S} I\} \quad \text{for all } S \subset \mathrm{Prim}(B).$$

b) The spectrum $\mathrm{Spec}(B)$ is the set of all unitary equivalence classes of irreducible representations of B, equipped with the smallest topology such that the map

$$\mathrm{Spec}(B) \ni [\rho] \mapsto \ker(\rho) \in \mathrm{Prim}(B),$$

is continuous.

Generally, the map $\mathrm{Spec}(B) \to \mathrm{Prim}(B)$ is surjective but not injective. We usually write simply ρ for the unitary equivalence class $[\rho]$ of an irreducible representation.

In the first part of this thesis we will encounter the following situation: Given a quotient $B_2 = B_1/I$ of a C*-algebra B_1 by an ideal I, we want to use our knowledge about $\mathrm{Spec}(B_2)$ to describe $\mathrm{Spec}(B_1)$. The following proposition comes in handy. For a reference see [10, pp. II.6.1.3, II.6.5.13].

Proposition 2.4.5. *Let $B_2 = B_1/I$ as above and let*

$$S := \{\rho \in \mathrm{Spec}(B_1) \mid \rho|_I = 0\}$$

be the set of (unitary equivalence classes of) irreducible representations of B_1 that vanish on the ideal I. Then S is a closed subset of $\mathrm{Spec}(B_1)$ and the map

$$\mathrm{Spec}(B_2) \ni \sigma \mapsto \sigma \circ \pi_I \in S$$

is a homeomorphism of topological spaces, where $\pi_I : B_1 \to B_2$ is the canonical quotient map.

2.5. Universal C*-algebras

A central notion in the first part of this thesis is the one of universal C*-algebras generated by some elements subject to certain relations. Given a family of symbols $(x_i)_{i\in I}$, a relation is any noncommutative polynomial p in the symbols x_i and x_i^*. Evidently, only finitely many symbols can feature in the polynomial and therefore we write $p = p(x_{i_1}, \dots, x_{i_k})$ if all symbols in p are from $\{x_{i_1}, \dots, x_{i_k}, x_{i_1}^*, \dots, x_{i_k}^*\}$.

Given a C^*-algebra B with elements $(X_i)_{i\in I} \subset B$ we say that the X_i satisfy the relation $p = p(x_{i_1}, \dots, x_{i_k})$ if the polynomial $p(X_{i_1}, \dots, X_{i_k})$ evaluates to zero in B. Usually we also express relations in words. For instance,

- "x is a projection" stands for the relations $x - x^*$ and $x^2 - x$,
- "x is a unitary" stands for the relations $xx^* - 1$ and $x^*x - 1$,
- "$\{x_1, \dots, x_n\}$ is a partition of unity" stands for the relations $\sum_{i=1}^n x_i - 1$ and $x_i^* - x_i, x_i^2 - x_i$ as well as $x_i x_j$ for all $i \neq j$.

Further, we also write $p = q$ for the relation $p - q$ if p and q are two noncommutative polynomials.

Definition 2.5.1. Let $(x_i)_{i\in I}$ be a family of symbols and let $(R_j)_{j\in J}$ be a family of noncommutative polynomials in x_i and x_i^*. Assume that A is a C*-algebra generated by elements $X_i \in A$ for $i \in I$ which satisfy the relations R_j for $j \in J$. We say that A is the universal C*-algebra generated by the elements $(x_i)_{i\in I}$ subject to the relations $(R_j)_{j\in J}$, written

$$A = C^*((x_i)_{i\in I} \mid (R_j)_{j\in J}),$$

if the following universal property holds: Whenever B is another C*-algebra generated by elements $Y_i \in B$ satisfying the relations R_j for all $j \in J$, then there is a unique $*$-homomorphism $\varphi : A \to B$ such that $\varphi(X_i) = Y_i$ for all $i \in I$.

Remark 2.5.2. Two remarks are in order: First, the universal C*-algebra $C^*((x_i)_{i\in I} \mid (R_j)_{j\in J})$ need not exist for arbitrary choices of generators and relations. In some situations, however, it is clear that it does exist. For instance, if the generators x_i must satisfy some bound $\|x_i\| \leq C_i$ in every model, then the universal C*-algebra is guaranteed to exist.

Second, if the universal C*-algebra exists it is unique up to isomorphism as an immediate consequence of the universal property. Thus, it is justified to speak of *the* universal C*-algebra; yet one must bear in mind that it is only defined up to isomorphism. We will deal with this situation as in the case of L^p functions: We will always assume that $C^*((x_i)_{i\in I} \mid (R_j)_{j\in J})$ is an arbitrary but fixed representative of the respective isomorphism class and write $B = C^*((x_i)_{i\in I} \mid (R_j)_{j\in J})$ to indicate that B is isomorphic to this universal C*-algebra.

Example 2.5.3. a) Every universal C*-algebra generated by projections, unitaries or isometries exists. This is due to the fact that these elements must satisfy the bound $\|x_i\| \leq 1$ in every C*-algebra.

b) The universal C*-algebra $C^*(u \mid uu^* = u^*u = 1)$ is isomorphic to $C(S^1)$, the C*-algebra of continuous functions on the unit circle.

c) The universal C*-algebra $C^*(u \mid u^n = 1, uu^* = u^*u = 1)$ is isomorphic to $\mathbb{C}^n$.

Part I.

C*-algebras generated by two partitions of unity

3. Classical results on two projections

The most simple, yet non-trivial, case of two partitions of unity is presented by two projections. Indeed, any projection p in a unital C*-algebra corresponds to the partition of unity $\{p, 1-p\}$. The general structure of two projections has been studied extensively in the literature. For concrete projection operators on a Hilbert space the situation is well-understood due to Halmos's Two Projection Theorem [45], while Pedersen was the first to give an explicit description of the universal C*-algebra generated by two projections [67]. We briefly review these results since they are the starting point for our investigation of bipartite graph C*-algebras. An extensive historical account can be found in [13].

Finally, we also review some generalizations of Halmos's and Pedersen's results to more than two projections.

3.1. Halmos's Two Projection Theorem and Pedersen's C*-algebra

Let us state Halmos's classical Two Projection Theorem.

Theorem 3.1.1 ([45, Theorem 2]). *Let P and Q be two projections on a Hilbert space $\mathcal{H}$ in* generic position *meaning that*

$$\begin{aligned} \operatorname{im}(P) \cap \operatorname{im}(Q) &= \{0\}, & \operatorname{im}(P) \cap \operatorname{im}(Q)^\perp &= \{0\}, \\ \operatorname{im}(P)^\perp \cap \operatorname{im}(Q) &= \{0\}, & \operatorname{im}(P)^\perp \cap \operatorname{im}(Q)^\perp &= \{0\}. \end{aligned}$$

Then P and Q are unitarily equivalent to operators of the form

$$\tilde{P} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \qquad \tilde{Q} = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix},$$

where C and S are positive contractions on some Hilbert space $\mathcal{K}$ satisfying $C^2 + S^2 = I$, I is the identity operator on $\mathcal{K}$.

The next theorem is Pedersen's description of the universal C*-algebra generated by two projections.

Theorem 3.1.2 ([67, Section 3]). *The universal unital C*-algebra generated by two projections p and q is isomorphic to the C*-algebra*

$$A := \{f \in C([0,1], M_2(\mathbb{C})) \mid f(0) \textit{ is diagonal}, f(1) \textit{ is diagonal}\}.$$

The isomorphism is given by sending p and q to the functions

$$\tilde{p}(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \tilde{q}(t) = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}.$$

We will later generalize Pedersen's theorem to a particular class of bipartite graph C*-algebras. Let us therefore discuss how the theorem can be proved. There are, in fact, a number of different approaches, and we try to give a selective overview of them. For the basic setup we follow the very lucid approach of Raeburn and Sinclair [72]. Perhaps ironically, we will not describe Pedersen's original proof because it is a slightly technical algebraic computation.

Proof. The universal property of $C^*(p,q)$ immediately yields a unital homomorphism from $C^*(p,q)$ into A which sends

$$p \mapsto \tilde{p}, \quad q \mapsto \tilde{q}.$$

Further, the elements $\tilde{p}$ and $\tilde{q}$ generate A (together with the unit). Indeed, for $t \in (0,1)$ the matrices $\tilde{p}(t)$ and $\tilde{q}(t)$ generate $M_2(\mathbb{C})$ as a C*-algebra, and at the boundary points $t = 0$ and $t = 1$ they generate the diagonal matrices. With a C*-algebra version of the Stone–Weierstrass Theorem [28, p. 11.1.8] one can conclude that A is generated by $\tilde{p}$ and $\tilde{q}$.

Thus, the only thing left to show is that the homomorphism is injective. For this it is sufficient that every irreducible representation of $C^*(p,q)$ factors through A. Let $\pi : C^*(p,q) \to B(\mathcal{H})$ be an irreducible representation. The proof requires that π is unitarily equivalent to either of the following maps:

$$\begin{aligned} &\pi_{00} : C^*(p,q) \to \mathbb{C},\ p \mapsto 0, q \mapsto 0, \qquad &&\pi_{01} : C^*(p,q) \to \mathbb{C},\ p \mapsto 0, q \mapsto 1, \\ &\pi_{10} : C^*(p,q) \to \mathbb{C},\ p \mapsto 1, q \mapsto 0, \qquad &&\pi_{11} : C^*(p,q) \to \mathbb{C},\ p \mapsto 1, q \mapsto 1, \end{aligned}$$

or

$$\pi_t : C^*(p,q) \to M_2, \quad p \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ q \mapsto \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix},$$

for some $t \in (0,1)$. A number of arguments can be used to prove this:

1. One method is to use that $\pi(p)$ and $\pi(q)$ are two projections on a Hilbert space which have no non-trivial jointly invariant subspace. If they are not in generic position, the latter entails $\pi = \pi_{ij}$ for some $i,j \in \{0,1\}$. Otherwise, one has up to unitary equivalence

$$\pi(p) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi(q) = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix},$$

   where C and S are positive contractions on some Hilbert space $\mathcal{K}$ with $C^2 + S^2 = I$. As C and S commute, the fact that π is irreducible implies $\mathcal{K} = \mathbb{C}$ and $\pi = \pi_t$ for some $t \in (0,1)$ follows.

2. A more elementary proof is suggested by Blackadar [10, p. IV.1.4.2]: It is not difficult to see that the corner $pC^*(p,q)p$ is commutative. Thus, it follows immediately that $\pi(p)$ is zero or one-dimensional. The same goes for $\pi(q)$, which implies that π is one- or two-dimensional. Now, one can list all projections on $\mathbb{C}$ and $\mathbb{C}^2$ up to unitary equivalence and conclude.

3. Raeburn and Sinclair [72] reduce the problem to the representation theory of groups. Indeed, $C^*(p,q)$ is also the group C*-algebra $C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$, and irreducible representations of $C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ are in 1-1 correspondence with unitary representations of $\mathbb{Z}_2 * \mathbb{Z}_2$. The group $\mathbb{Z}_2 * \mathbb{Z}_2$ is isomorphic to the semidirect product $\mathbb{Z} \rtimes \mathbb{Z}_2$ where the action is implemented by multiplication with ± 1. One can use the Mackey machine to determine the unitary representations of $\mathbb{Z} \rtimes \mathbb{Z}_2$ and conclude.

4. Another approach is suggested e.g. by an exercise in [89]: One can show that the element $x = 1 - p - q + pq + qp$ is in the center of $C^*(p,q)$. Indeed, one observes
$$px - xp = pqp - pqp = 0,$$
and similarly $qx - xq = 0$. Then a dense subset of $C^*(p,q)$ is spanned by elements in $C^*(x)\{p, q, qp, pq\}$. Evidently, $\pi(C^*(x)) \cong \mathbb{C}$, and it follows that $\pi(C^*(p,q))$ is spanned by $\pi(p), \pi(q), \pi(pq)$ and $\pi(qp)$. In particular, $\pi(C^*(p,q))$ is at most four-dimensional as a vector space. As π is also irreducible it follows $\pi(C^*(p,q)) \cong \mathbb{C}$ or $\pi(C^*(p,q)) \cong M_2(\mathbb{C})$. In any event π is one- or two-dimensional and one can conclude as above.

5. Finally, there is also a much more indirect way to prove that the irreducible representations of $C^*(p,q)$ are one- or two-dimensional. This involves proving that for every four elements $x_1, x_2, x_3, x_4 \in C^*(p,q)$ one has
$$\sum_{\sigma \in S_4} \operatorname{sgn}(\sigma) x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)} = 0.$$
As a general fact, this condition implies that the C*-algebra $C^*(p,q)$ is 2-subhomogeneous (see e.g. [10, p. IV.1.4.6]). One can assume without loss of generality that the x_i are finite products of p and q and check the above identity by combinatorial arguments. Then, one concludes as above. This approach has been used by Roch and Silbermann [75] to investigate the more general case of Banach algebras generated by two idempotents.

□

3.2. Generalizations

By a result of Davis from 1955, three projections suffice to generate the algebra of bounded operators $B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ as a von Neumann algebra [26]. Thus, there is generally no hope to generalize Halmos's or Pedersen's results to an arbitrary number of projections. Yet, other generalizations have been considered in the literature which motivate some results of this thesis. We briefly review these results here with a clear focus on connections to our later work. For a more extensive overview we refer to [13].

Motivated by applications in the theory of singular integral operators, Roch and Silbermann generalized Halmos's/Pedersen's results to the situation of two idempotents in a Banach algebra [75]. Together with a number of coauthors, they further developed what they called an N-Projection Theorem which describes the structure of Banach algebras generated by one idempotent P as well as N further idempotents $Q_1, \dots, Q_N$ with $Q_i Q_j = \delta_{ij} Q_i$ and $Q_1 + \dots + Q_n$ satisfying certain additional relations [12]. Thus, the corresponding C*-algebras are generated by two partitions of unity containing 2 and N projections, respectively. However, the additional relations imposed on these two partitions of unity are quite special (they are motivated by singular integral operators) and are not comparable with the orthogonality relations required for bipartite graph C*-algebras.

In the context of Banach algebras, the results are usually formulated in the form of a symbol calculus, and we will not try to reproduce them here. However, in [86] Vasilevski, one of the authors of the aforementioned N-Projection Theorem, investigated the particular special case where the Banach algebra is a C*-algebra and the idempotents are projections. We will later find similar results for bipartite graph C*-algebras.

Vasilevski investigates projections P and $Q_1, \dots, Q_n$ on a Hilbert space $\mathcal{H}$, where

$$Q_1 + \dots + Q_n = I,$$

and additionally

$$\operatorname{im}(P) \cap \ker(Q_i) = \{0\} = \ker(P) \cap \operatorname{im}(Q_i) \tag{3.1}$$

holds for all $i \leq n$. The latter condition can be seen as a generalization of Halmos's "generic position".

The problem is to describe the C*-algebra $C^*(P, Q_1, \dots, Q_n)$ generated by these projections. For this, consider the subspaces $M_i \subset \mathcal{H}$ associated to the projections Q_i. The M_i form a partition of $\mathcal{H}$, so with respect to $\mathcal{H} \cong \bigoplus_i M_i$ we can write P (and the Q_i) in block matrix form:

$$P = (Q_i P Q_j)_{i,j=1}^n, \quad Q_i = \operatorname{diag}(0, \dots, I_{M_i}, \dots, 0).$$

Moreover, using (3.1) one obtains that the M_i all have the same dimension, so that they may be identified with a single Hilbert space $\mathcal{K}$. This leads to the following generalization of Halmos's Two Projection Theorem.

Theorem 3.2.1 ([86, Section 2])**.** *Let P and $Q_1, \dots, Q_n$ be as above. There are positive operators $D_1, \dots, D_n$ on some Hilbert space $\mathcal{K}$ such that up to unitary equivalence one has*

$$P = (D_i D_j^*)_{i,j=1}^n, \quad Q_i = \operatorname{diag}(0, \dots, I_{\mathcal{K}}, \dots, 0).$$

These operators satisfy

$$D_1 D_1^* + \cdots + D_n D_n^* = I_{\mathcal{K}}.$$

Conversely, any such family of operators $(D_i)_{i=1}^n$ *gives rise to projections* P *and* $(Q_i)_{i=1}^n$ *as above.*

Further, the positive contractions $(D_i)_{i=1}^n$ *determine the projections* P *and* $(Q_i)_{i=1}^n$ *up to unitary equivalence, i.e. two such families of contractions are unitarily equivalent if and only if the associated projections are unitarily equivalent.*

It is generally not possible to use the above to derive a corresponding generalization of Pedersen's theorem. This is due to the fact that in the special situation $n = 2$ all the positive contractions D_i commute. Still, Vasilevski obtains an explicit description of the universal C*-algebra generated by projections P and $Q_1, \ldots, Q_n$ as above with the additional assumption that the $D_1, \ldots, D_n$ commute.

4. Bipartite graph C*-algebras

In this section, we introduce the main object of our investigations: a class of C*-algebras that are generated by two universal finite partitions of unity which satisfy certain orthogonality relations. The precise relations are given by a bipartite graph, and this is why we call the obtained C*-algebras “bipartite graph C*-algebras”.

4.1. Bipartite graphs

First, let us recall the notion of a bipartite graph and settle some notation. Throughout the following all graphs are finite.

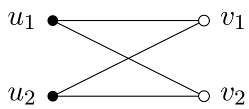
Definition 4.1.1. A bipartite graph G consists of two (finite) vertex sets U and V together with an edge set $E \subset \{\{u, v\} \mid u \in U, v \in V\}$. We write $G = (U, V, E)$.

For two vertices in a bipartite graph $G = (U, V, E)$ let us write $u \sim v$ if $\{u, v\} \in E$ and let $\mathcal{N}(u) := \{v \in V \mid v \sim u\}$ ($\mathcal{N}(v) := \{u \in U \mid v \sim u\}$) be the neighbors of u (v). Since an edge $e = \{u, v\}$ is a two-element set, we write $u \in e$ to denote that u is one of its endpoints. A path μ in a bipartite graph G is a finite sequence of vertices $v_1 \dots v_n$ such that $v_i \sim v_{i+1}$ holds for all $i < n$. For every $i < n$, we say that v_i is contained in μ. We set $s(\mu) = v_1$ and $r(\mu) = v_n$. For two paths $\mu = v_1 \dots v_m$ and $\nu = u_1 \dots u_n$ with $r(\mu) = s(\nu)$ we write $\mu\nu$ for the combined path $v_1 \dots v_m u_1 \dots u_n$.

Example 4.1.2. For $m, n \in \mathbb{N}$, the complete bipartite graph $K_{m,n} = (U, V, E)$ of order (m, n) is given by

$$\begin{aligned} U &= \{u_1, \dots, u_m\}, \\ V &= \{v_1, \dots, v_n\}, \\ E &= \{\{u_i, v_j\} \mid i = 1, \dots, m, j = 1, \dots, n\}. \end{aligned}$$

In particular, the graph $K_{2,2}$ is described by the sketch below.



Definition 4.1.3. Let $G = (U, V, E)$ and $G' = (U', V', E')$ be bipartite graphs.

1. G' is a subgraph of G, written $G' \subset G$, if

- $U' \subset U$,
- $V' \subset V$, and
- $E' \subset \{\{u, v\} \in E \mid u \in U', v \in V'\}$,

or if the same holds after swapping U' and V'.

2. G' is the subgraph of G induced by the set $\{x_1, \ldots, x_n\} \subset U \cup V$, written $G' = G(x_1, \ldots, x_n)$ if
   - $U' = U \cap \{x_1, \ldots, x_n\}$,
   - $V' = V \cap \{x_1, \ldots, x_n\}$, and
   - $E' = \{e \in E \mid e \subset \{x_1, \ldots, x_n\}\}$.

   Similarly, G' is the subgraph of G induced by the set $\{e_1, \ldots, e_n\} \subset E$, written $G' = G(e_1, \ldots, e_n)$ if
   - $U' = U \cap (\bigcup\{e_1, \ldots, e_n\})$,
   - $V' = V \cap (\bigcup\{e_1, \ldots, e_n\})$,
   - $E' = \{e_1, \ldots, e_n\}$.
3. G' is isomorphic to G, written $G' \cong G$, if there are two bijective maps $\varphi : U \to U'$ and $\psi : V \to V'$ such that
$$E' = \{\{\varphi(u), \psi(v)\} \mid u \in U, v \in V, \{u, v\} \in E\},$$
   or if the same holds after swapping U' and V'.

4.2. Definition and first properties

Let us now introduce bipartite graph C*-algebras.

Definition 4.2.1. Given a bipartite graph $G = (U, V, E)$ let $C^*(G)$ be the universal C^*-algebra generated by a family of projections $(p_x)_{x \in U \cup V}$ subject to the following relations:

$$\sum_{u \in U} p_u = 1 = \sum_{v \in V} p_v, \tag{GP1}$$
$$p_u p_v = 0 \ \text{ if } \{u, v\} \notin E. \tag{GP2}$$

We call a family of projections $(P_x)_{x \in U \cup V} \subset B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ a G-projection family if they satisfy the relations (GP1) and (GP2).

In view of Remark 2.5.2 the universal C*-algebra $C^*(G)$ exists for any bipartite graph G.

We warn the reader that, despite the name, bipartite graph C*-algebras are not directly related to the well-known graph C*-algebras. Whereas graph C*-algebras are universally generated by partial isometries, bipartite graph C*-algebras are universally generated by projections. This leads to a very different behaviour of the two classes of C*-algebras.

Example 4.2.2. Consider the complete bipartite graph $K_{2,2}$ of order $(2,2)$. Then $C^*(K_{2,2})$ is isomorphic to the universal unital C^*-algebra $C^*(p,q)$ generated by two projections. Indeed, the two left vertices of $K_{2,2}$ are associated to the projections p and $1-p$, while the two right vertices are associated to the projections q and $1-q$. As $K_{2,2}$ is a complete graph there are no orthogonality requirements on these two partitions of unity. More generally, we have $C^*(K_{n,m}) \cong \mathbb{C}^n *_{\mathbb{C}} \mathbb{C}^m$ for all $n,m \in \mathbb{N} \setminus \{0\}$.

Using the relations (GP1) and (GP2) one easily sees that a dense subset of $C^*(G)$ is spanned by elements associated to paths in G.

Proposition 4.2.3. *Let G be a bipartite graph as in Definition 4.2.1, and for every path $\mu = x_1 \dots x_n$ in G, write $p_\mu := p_{x_1} \cdots p_{x_n}$ for the associated element in $C^*(G)$. Then the elements p_μ span a dense subset of $C^*(G)$.*

Proof. Evidently, a dense subset of $C^*(G)$ is spanned by arbitrary products of the form $a = p_{x_1} p_{x_2} \cdots p_{x_n}$ with $x_1, \dots, x_n \in U \cup V$. Since the p_{x_i} are projections we may assume without loss of generality that $x_i \neq x_{i+1}$ holds for all $i < n$.

To prove the statement, it suffices to observe that a vanishes whenever $x_1 \dots x_n$ is not a path in G. Indeed, whenever x_i and x_{i+1} for $i < n$ are both in U or both in V, then $p_{x_i} p_{x_{i+1}} = 0$, for $x_i \neq x_{i+1}$ (by assumption) and $(p_u)_{u \in U}$ (resp. $(p_v)_{v \in V}$) is a family of pairwise orthogonal projections by (GP1). Thus, a vanishes if the x_i are not alternatingly from U and V. Finally, assume without loss of generality that $x_i \in U$ and $x_{i+1} \in V$ for $i < n$. Then $p_{x_i} p_{x_{i+1}} = 0$ unless $\{x_i, x_{i+1}\} \in E$ by (GP2). This proves the claim. ☐

In a similar way as the last proposition, one obtains the next lemma which will be useful later.

Lemma 4.2.4. *Let $G = (U, V, E)$ be a bipartite graph, and let $x_1, x_2, y \in U \cup V$ be three distinct vertices. Then*

$$p_{x_1} p_y p_{x_2} \neq 0 \quad \implies \quad \{y\} \subsetneq \mathcal{N}(x_1) \cap \mathcal{N}(x_2).$$

In particular, in this case there must be a fourth vertex y' such that

$$G(x_1, x_2, y, y') \cong K_{2,2}.$$

Proof. Assume $p_{x_1} p_y p_{x_2} \neq 0$ and assume that the statement on the right-hand side is not true, i.e. $\mathcal{N}(x_1) \cap \mathcal{N}(x_2)$ contains at most the vertex y. If y is not contained, then one has directly $p_{x_1} p_y = 0$ or $p_y p_{x_2} = 0$ thanks to (GP2) and the fact that x_1, y, x_2 are distinct. Otherwise, assume without loss of generality $x_1, x_2 \in V$ and $y \in U$. Then one observes

$$p_{x_1} p_y p_{x_2} = p_{x_1} \left(\sum_{u \in U} p_u \right) p_{x_2} = p_{x_1} p_{x_2} = 0.$$

Indeed, for all $u \in U \setminus \{y\}$ we have $u \notin \mathcal{N}(x_1)$ or $u \notin \mathcal{N}(x_2)$, since by assumption y is the only possible common neighbour of x_1 and x_2 in U. Hence, for these u (GP2) yields $p_{x_1} p_u p_{x_2} = 0$. The latter equality follows directly from (GP1) since $\sum_{u \in U} p_u = 1$. Finally, we have $p_{x_1} p_{x_2} = 0$, for $x_1 \neq x_2$ and $(p_v)_{v \in V}$ is a family of pairwise orthogonal projections by (GP1). □

Let us discuss how bipartite graph C*-algebras behave with respect to subgraphs.

Proposition 4.2.5. *Let $G = (U, V, E)$ be a bipartite graph and let $H = (U', V', E')$ be a subgraph induced by a set of vertices, i.e.*

$$H = G(x_1, \ldots, x_n)$$

for some subset $\{x_1, \ldots, x_n\} = U' \cup V'$ of $U \cup V$. Then $C^*(H) \cong C^*(G)/(S)$, where $(S) \subset C^*(G)$ is the ideal generated by the elements p_x with $x \notin U' \cup V'$.

Proof. Let $(\hat{p}_x)_{x \in U' \cup V'}$ be the generators of $C^*(H)$ and set

$$P_x := \begin{cases} \hat{p}_x, & \text{if } x \in U' \cup V', \\ 0, & \text{otherwise,} \end{cases}$$

for all $x \in U \cup V$. Then $(P_x)_{x \in U \cup V}$ is a G-projection family, and the universal property of $C^*(G)$ yields a $*$-homomorphism $\varphi : C^*(G) \to C^*(H)$ such that $\varphi(p_x) = P_x$ for all $x \in U \cup V$. Evidently, φ is surjective and one has $(S) \subset \ker(\varphi)$. Therefore, φ induces a surjective $*$-homomorphism $\tilde{\varphi} : C^*(G)/(S) \to C^*(H)$ which maps $[p_x]$ to $\hat{p}_x$ for all $x \in U' \cup V'$ where $[p_x]$ is the image of $p_x \in C^*(G)$ under the quotient map.

The universal property of $C^*(H)$ yields an inverse $*$-homomorphism. Indeed, one has $[p_x] = 0$ whenever $x \notin U' \cup V'$. Therefore, the families $([p_x])_{x \in U'}$ and $([p_x])_{x \in V'}$ are partitions of unity in $C^*(G)/(S)$. What's more, these projections satisfy the orthogonality relations (GP2) for $C^*(H)$ since H is an induced subgraph of G, i.e. every non-edge in H is also a non-edge in G. Thus, we obtain a unique $*$-homomorphism $\tilde{\psi} : C^*(H) \to C^*(G)/(S)$ with $\tilde{\psi}(\hat{p}_x) = [p_x]$ for all $x \in U' \cup V'$. Clearly $\tilde{\psi}$ and $\tilde{\varphi}$ are inverse to each other. □

Corollary 4.2.6. *Let G be a bipartite graph with $K_{2,3} \subset G$. Then $C^*(G)$ is neither nuclear nor exact.*

Proof. Combining the previous Proposition 4.2.5 and Example 4.2.2, we see that the bipartite graph C*-algebra $\mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^3$ is a quotient of $C^*(G)$ as soon as $K_{2,3} \subset G$ holds. It is well-known that $\mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^3$ is neither nuclear nor exact and both properties are preserved under taking quotients. Therefore, $C^*(G)$ is neither nuclear nor exact as well. □

We end the section with a technical looking lemma that will be useful later in the classification of bipartite graph C*-algebras.

Lemma 4.2.7. *Let $G = (U, V, E)$ be a bipartite graph and assume that $e = \{u, v\} \in E$ is not contained in a subgraph of G that is isomorphic to $K_{2,2}$. Then, we have*

$$C^*(G) \cong C^*(G') \oplus \mathbb{C},$$

where G' is obtained by deleting the edge e from G. The isomorphism sends $p_x \in C^*(G)$ to the corresponding element $(p_x, 0) \in C^*(G') \oplus \mathbb{C}$ if $x \notin e$, and otherwise to $(p_x, 1)$.

Proof. In view of Lemma 4.2.4 one has for all $v' \in V \setminus \{v\}$ and $u' \in U \setminus \{u\}$

$$p_v p_u p_{v'} = 0, \quad p_u p_v p_{u'} = 0.$$

Using this it is not hard to show that $p_u p_v = p_v p_u$ is a projection, and it is orthogonal to all p_x with $x \notin \{u, v\}$. Further, it is easily checked that the family $(p'_x)_{x \in U \cup V}$ defined by

$$p'_x := \begin{cases} p_x, & \text{if } x \notin e, \\ p_x - p_u p_v, & \text{if } x \in e, \end{cases}$$

is a universal G'-projection family in $C^*(G)$. Thus, it follows

$$C^*(G) \cong C^*(p'_x \mid x \in U \cup V) \oplus \mathbb{C} p_u p_v \cong C^*(G') \oplus \mathbb{C},$$

where the isomorphism maps $p_u p_v \mapsto (0, 1)$. $\square$

4.3. Alternative generators of bipartite graph C*-algebras

In the previous section, we introduced bipartite graph C^*-algebras $C^*(G)$ as universal C^*-algebras which are generated by projections associated to the vertices of a bipartite graph G. Interestingly, one can define $C^*(G)$ in a different way as universal C^*-algebra generated by contractions associated to the edges of G. This is the content of the following proposition.

Proposition 4.3.1. *Let $G = (U, V, E)$ be a bipartite graph. Then $C^*(G)$ is the universal C^*-algebra generated by a family of elements $(x_e)_{e \in E}$ satisfying*

$$x_e^* x_f = 0 \qquad \text{if } e \cap f \cap U = \emptyset, \tag{GC1}$$

$$x_e x_f^* = 0 \qquad \text{if } e \cap f \cap V = \emptyset, \tag{GC2}$$

$$\left(\sum_{e \in E} x_e^*\right) x_f = x_f, \tag{GC3}$$

$$x_e \left(\sum_{f \in E} x_f^*\right) = x_e, \tag{GC4}$$

for all edges $e, f \in E$, respectively. In particular, the x_e are contractions which satisfy $x_e x_e^* x_e = x_e^2$.

The relation $xx^*x = x^2$ in the statement means that x is a product of two projections, see e.g. [70, Theorem 8].

Proof. First of all, conditions (GC1) and (GC2) together with (GC3) entail for every edge $e = \{u, v\} \in E$ that

$$x_e x_e^* x_e = x_e \left(\sum_{f \in E} x_f^* \right) x_e = x_e^2,$$

since in the second expression all terms for $f \neq e$ vanish. It follows in particular that the x_e are contractions, and the universal C^*-algebra A generated by elements $(x_e)_{e \in E}$ satisfying (GC1)–(GC4) exists. Further, (GC3) and (GC4) imply that A is unital with unit $1 = \sum_{e \in E} x_e^*$.

Let us use the universal property of $C^*(G)$ to find a $*$-homomorphism

$$\varphi : C^*(G) \to A$$

with

$$\varphi(p_x) = \sum_{e \in E : x \in e} x_e =: P_x \quad \text{for all } x \in U \cup V.$$

First, one checks $P_x = P_x^*$. Indeed, assume $x \in U$ and apply (GC3) and (GC1) to obtain

$$\begin{aligned} P_x^* = \sum_{e \in E : x \in e} x_e^* = \sum_{e \in E : x \in e} x_e^* \left(\sum_{f \in E} x_f \right) &= \sum_{e \in E} x_e^* \left(\sum_{f \in E : x \in f} x_f \right) \\ &= \sum_{f \in E : x \in f} \left(\sum_{e \in E} x_e^* \right) x_f = P_x. \end{aligned}$$

In the second step, we use (GC3) after taking the involution on both sides. If $x \in V$, one can use (GC4) and (GC2) to obtain in a similar way

$$\begin{aligned} P_x^* = \sum_{e \in E : x \in e} x_e^* = \sum_{e \in E : x \in e} \left(\sum_{f \in E} x_f \right) x_e^* &= \sum_{e \in E} \left(\sum_{f \in E : x \in f} x_f \right) x_e^* \\ &= \sum_{f \in E : x \in f} x_f \left(\sum_{e \in E} x_e^* \right) = P_x. \end{aligned}$$

Next, let us prove $P_x^2 = P_x$. Again we first assume $x \in U$. Then (GC1) and (GC3) yield

$$P_x^2 = \left(\sum_{e \in E : x \in e} x_e^* \right) \left(\sum_{f \in E : x \in f} x_f \right) = \sum_{f \in E : x \in f} \left(\sum_{e \in E} x_e^* \right) x_f = P_x.$$

If $x \in V$, then one can use (GC2) and (GC4) to obtain

$$P_x^2 = \left(\sum_{e \in E : x \in e} x_e \right) \left(\sum_{f \in E : x \in f} x_f^* \right) = \sum_{f \in E : x \in f} x_f \left(\sum_{e \in E} x_e^* \right) = P_x.$$

Since

$$\begin{aligned} \sum_{u \in U} P_u &= \sum_{u \in U} \sum_{e \in E : u \in e} x_e^* = \sum_{e \in E} x_e^* = 1, \text{ and} \\ \sum_{v \in V} P_v &= \sum_{v \in V} \sum_{e \in E : v \in e} x_e^* = \sum_{e \in E} x_e^* = 1, \end{aligned}$$

the projections $(P_x)_{x \in U \cup V}$ satisfy the relations (GP1). It remains to check the orthogonality relations (GP2). For this, let $u \in U$ and $v \in V$ be two vertices such that $\{u, v\} \notin E$. Then one has

$$\begin{aligned} P_v P_u &= \left(\sum_{e \in E : v \in e} x_e \right) \left(\sum_{g \in E : u \in g} x_g \right) \\ &= \left(\sum_{e \in E : v \in e} x_e \right) \left(\sum_{f \in E} x_f^* \right) \left(\sum_{g \in E : u \in g} x_g \right) \\ &= \sum_{e, f, g \in E : v \in f, u \in f} x_e x_f^* x_g \\ &= 0, \end{aligned}$$

where the second-last equality follows from (GC1) and (GC2), and the last equality follows from the fact that u and v are not connected by an edge in G. This proves (GP2). Altogether, we showed that the elements $P_x \in A$ for $x \in U \cup V$ are projections satisfying the relations (GP1) and (GP2). Thus, the universal property of $C^*(G)$ yields the desired $*$-homomorphism $\varphi : C^*(G) \to A$.

Conversely, one can use the universal property of A to obtain an inverse $*$-homomorphism $\psi : A \to C^*(G)$ with

$$\psi(x_e) = p_u p_v =: X_e \qquad \text{for all } e = \{u, v\} \in E,$$

where we implicitly require $u \in U$ and $v \in V$. We need to check that the elements $X_e \in C^*(G)$ satisfy the relations (GC1)–(GC4). For (GC1) let $e = \{u_1, v_1\}$ and $f = \{u_2, v_2\}$ with $u_1, u_2 \in U$ and $v_1, v_2 \in V$ be two edges such that $u_1 \neq u_2$. Then one has

$$X_e^* X_f = p_{v_1} p_{u_1} p_{u_2} p_{v_2} = 0$$

since $p_{u_1} p_{u_2} = 0$ by (GP1). The relation (GC2) is checked in a similar way. For (GC3) let $f = \{u, v\} \in E$ and observe

$$\begin{aligned}\left(\sum_{e\in E} X_e^*\right) X_f &= \left(\sum_{e\in E} p_{v_e}p_{u_e}\right) p_u p_v \\ &= \left(\sum_{u'\in U, v'\in V} p_{v'}p_{u'}\right) p_u p_v \\ &= \left(\sum_{v'\in V} p_{v'}\right)\left(\sum_{u'\in U} p_{u'}\right) p_u p_v \\ &= p_u p_v \\ &= X_f,\end{aligned}$$

where we write $e = \{u_e, v_e\}$ for all edges $e \in E$ with $u_e \in U$ and $v_e \in V$. In the second step, we use that by (GP2) all superfluous terms in the sum vanish, and in the fourth step we use (GP1). Relation (GC4) is checked analogously.

Finally, it remains to verify that ψ and φ are inverse to each other. For $e = \{u, v\} \in E$ observe

$$\begin{aligned}\varphi(\psi(x_e)) = \varphi(p_u p_v) = \varphi(p_u)\varphi(p_v) &= \left(\sum_{f\in E: u\in f} x_f^*\right)\left(\sum_{g\in E: v\in g} x_g\right) \\ &= \left(\sum_{f\in E: u\in f} x_f^*\right) x_e = \left(\sum_{f\in E} x_f^*\right) x_e = x_e,\end{aligned}$$

where we use (GC1) in the forth and fifth step, and (GC3) in the last step. Further, one has for $u \in U$

$$\psi(\varphi(p_u)) = \psi\left(\sum_{e\in E: u\in e} x_e\right) = \sum_{e\in E: u\in e} p_u p_{v_e} = p_u\left(\sum_{v'\in V} p_{v'}\right) = p_u,$$

where we use (GP2) in the second-last step and (GP1) in the last step. One checks $\psi(\varphi(p_v)) = p_v$ for $v \in V$ in a similar way. This proves that φ and ψ are inverse to each other, and hence they yield the desired isomorphism $C^*(G) \cong A$. □

4.4. Projections in generic position

Throughout this section let $G = (U, V, E)$ be a connected bipartite graph and let $(P_x)_{x\in U\cup V}$ be a G-projection-family on some Hilbert space $\mathcal{H}$ as in Definition 4.2.1. Further, set

$$L_x := \operatorname{im}(P_x) \subset \mathcal{H}$$

for all $x \in U \cup V$. Using arguments similar to those of Vasilevski [86], we obtain a generalization of Halmos's Two Projection Theorem (Theorem 3.1.1) to G-projection families.

Definition 4.4.1. We say that the P_x are in *generic position* if

$$L_u \cap L_v^{\perp} = \{0\} = L_u^{\perp} \cap L_v$$

holds for all edges $\{u, v\} \in E$.

Lemma 4.4.2. *Assume that the P_x are in generic position. Then one has*

$$L_x \cong L_y$$

for all $x, y \in U \cup V$.

Proof. We borrow the proof from [45, Theorem 1]. As G is connected it suffices to show that claim for $\{x, y\} = \{u, v\} \in E$. So let $u \in U$ and $v \in V$ be two adjacent vertices. We claim that the operator $P_u|_{L_v} : L_v \to L_u$ is injective with dense range. Indeed, for any $f \in L_v$ one has

$$\begin{aligned} P_u f = 0 &\implies f \in L_v \cap L_u^{\perp} \\ &\implies f = 0, \end{aligned}$$

which proves injectivity. Next, let $g \in L_u$ and assume that $g \perp P_u f$ holds for all $f \in L_v$. Then one has for all $f \in L_v$

$$0 = \langle g, P_u f\rangle = \langle P_u g, f\rangle = \langle g, f\rangle,$$

and hence we see $g \in L_u \cap L_v^{\perp}$ which implies $g = 0$. Thus, the operator $P_u|_{L_v}$ has dense range in L_u. Consequently, the spaces L_u and L_v are isometrically isomorphic. □

Theorem 4.4.3. *Let $(P_x)_{x \in U \cup V}$ be a G-projection family on a Hilbert space $\mathcal{H}$ in generic position where G is connected. Then there are operators $(C_{uv})_{u \in U, v \in V}$ on a Hilbert space $\mathcal{K}$ such that up to unitary equivalence one has*

$$\begin{aligned} P_u &= (C_{uv_1}^* C_{uv_2})_{v_1, v_2 \in V} \in M_V(\mathcal{B}(\mathcal{K})), \\ P_v &= (\delta_{v_1 v} \delta_{v_2 v})_{v_1, v_2 \in V} \in M_V(\mathcal{B}(\mathcal{K})), \end{aligned} \tag{4.1}$$

for all $u \in U$ and $v \in V$, where $\delta_{v_1 v} = I \in B(\mathcal{K})$ if $v_1 = v$ and $\delta_{v_1 v} = 0$ otherwise, and $M_V(\mathcal{B}(\mathcal{K}))$ is the algebra of square matrices indexed by V with entries from the bounded operators on $\mathcal{K}$.

The operator C_{uv} vanishes if $\{u, v\} \notin E$, and is injective with dense range otherwise. Moreover, the operators $(C_{uv})_{u \in U, v \in V}$ satisfy

$$\sum_{u \in U} C_{uv_1}^* C_{uv_2} = \delta_{v_1 v_2}, \tag{4.2}$$

$$\sum_{v \in V} C_{u_1 v} C_{u_2 v}^* = \delta_{u_1 u_2}. \tag{4.3}$$

Conversely, every family of operators $(C_{uv})_{u \in U, v \in V}$ with the above properties gives rise to a G-projection family $(P_x)_{x \in U \cup V}$ via the formula (4.1).

Proof. Assume that $(P_x)_{x\in U\cup V}$ is a G-projection family on the Hilbert space $\mathcal{H}$. With respect to the decomposition $\mathcal{H} = \bigoplus_{v\in V} L_v$ we can write the P_x in block matrix form as

$$P_u = (P_{v_1} P_u P_{v_2})_{v_1,v_2\in V}, \quad \text{and} \quad P_v = (\delta_{v_1 v}\delta_{v_2 v})_{v_1,v_2\in V}.$$

By Lemma 4.4.2 there is a Hilbert space $\mathcal{K}$ and a family of isometric isomorphisms $U_x : L_x \to \mathcal{K}$ with $x \in U \cup V$. Let $\mathcal{U} := \operatorname{diag}((U_v)_{v\in V}) \in M_V(B(\mathcal{K}))$ and observe

$$\mathcal{U}P_u\mathcal{U}^* = (U_{v_1} P_{v_1} P_u U_u^* U_u P_u P_{v_2} U_{v_2}^*)_{v_1,v_2\in V}, \quad \text{and} \quad \mathcal{U}P_v\mathcal{U}^* = (\delta_{v_1 v}\delta_{v_2 v})_{v_1,v_2\in V}.$$

Setting

$$C_{uv} := U_u P_u P_v U_v^* \in B(\mathcal{K})$$

for all $u \in U$ and $v \in V$ we immediately obtain (4.1).

Let us prove that the operators C_{uv} have the desired properties. If $\{u, v\} \notin E$ is not an edge then (GP2) entails $P_u P_v = 0$ and thus $C_{uv} = 0$. Otherwise, C_{uv} is injective with dense range by the proof of Lemma 4.4.2. Moreover, for every $u_1, u_2 \in U$ and $v_1, v_2 \in V$ one easily checks

$$\begin{aligned}
\sum_{u\in U} C_{uv_1}^* C_{uv_2} &= \sum_{u\in U} U_{v_1} P_{v_1} P_u U_u^* U_u P_u P_{v_2} U_{v_2}^* \\
&= \sum_{u\in U} U_{v_1} P_{v_1} P_u P_{v_2} U_{v_2}^* \\
&= U_{v_1} P_{v_1} P_{v_2} U_{v_2}^* \\
&= \delta_{v_1 v_2},
\end{aligned}$$

using (GP1), and

$$\begin{aligned}
\sum_{v\in V} C_{u_1 v} C_{u_2 v}^* &= \sum_{v\in V} U_{u_1} P_{u_1} P_v U_v^* U_v P_v P_{u_2} U_{u_2}^* \\
&= U_{u_1} P_{u_1} P_{u_2} U_{u_2}^* \\
&= \delta_{u_1 u_2}
\end{aligned}$$

using (GP1) again.

In the other direction, assume that $(C_{uv})_{u\in U, v\in V}$ is a family of operators on some Hilbert space $\mathcal{K}$ satisfying the conditions from the statement. Let operators P_x on the Hilbert space $\mathcal{K}^V$ be defined via (4.1). We need to show that the P_x form a G-projection family. Evidently, the P_v with $v \in V$ are pairwise orthogonal projections adding up to the unit. On the other hand, the P_u with $u \in U$ are clearly selfadjoint, and we have for every $v_1, v_2 \in V$ by (4.3)

$$\begin{aligned}
[P_{u_1} P_{u_2}]_{v_1 v_2} &= \sum_{v_3\in V} C_{u_1 v_1}^* C_{u_1 v_3} C_{u_2 v_3}^* C_{u_2 v_2} \\
&= C_{u_1 v_1}^* \left(\sum_{v_3\in V} C_{u_1 v_3} C_{u_2 v_3}^* \right) C_{u_2 v_2} \\
&= \delta_{u_1 u_2} C_{u_1 v_1}^* C_{u_2 v_2} \\
&= \delta_{u_1 u_2} [P_{u_1}]_{v_1 v_2}.
\end{aligned}$$

Further, using (4.2) one easily obtains $\sum_{u \in U} P_u = I \in M_V(B(\mathcal{K}))$. Hence, the P_u with $u \in U$ form a partition of unity as well. Finally, assume that $\{u, v\}$ with $u \in U$ and $v \in V$ is not an edge in G. Then it is not hard to check that every entry of P_u in the v-column or v-row is equal to zero, which entails $P_u P_v = 0$, that is, P_u and P_v are orthogonal. Therefore, the P_x form a G-projection family.

It only remains to show that the P_x are in generic position. Let $\{u, v\} \in E$ be an edge. First, we show $L_u^\perp \cap L_v = \{0\}$. Assume $g = (g_{v_1})_{v_1 \in V} \in L_u^\perp \cap L_v \subset \mathcal{K}^V$. Then $g_{v_1} = 0$ for all $v_1 \neq v$ as $g \in L_v$. Thus, one checks

$$P_u g = \left(C_{uv_1}^* C_{uv} g_v\right)_{v_1 \in V}$$

which vanishes because $g \in L_u^\perp$. In particular, it follows that

$$\sum_{v_1 \in V} C_{uv_1} C_{uv_1}^* C_{uv} g_v = C_{uv} g_v = 0.$$

Since C_{uv} is injective, we conclude $g_v = 0$ and hence $g = 0$.

It remains to show $L_u \cap L_v^\perp = \{0\}$. Let $g = (g_{v_1}) \in L_u \cap L_v^\perp \subset \mathcal{K}^V$. Then one has $g_v = 0$ as $g \in L_v^\perp$. Further, since $g \in L_u$ we have $P_u g = g$, i.e.

$$0 = (P_u g)_v = \sum_{v_1 \in V} C_{uv}^* C_{uv_1} g_{v_1} = C_{uv}^* \left(\sum_{v_1 \in V} C_{uv_1} g_{v_1} \right).$$

Consequently, for $f := \sum_{v_1 \in V} C_{uv_1} g_{v_1}$ we have $C_{uv}^* f = 0$. As C_{uv} has dense range it follows $f = 0$. Thus, we obtain for every $v_2 \in V$

$$0 = C_{uv_2}^* f = \sum_{v_1 \in V} C_{uv_2}^* C_{uv_1} g_{v_1} = (P_u g)_{v_2} = g_{v_2}.$$

We conclude $g = 0$ as desired. $\square$

5. Hypergraph C*-algebras

The motivation for studying bipartite graph C*-algebras comes from hypergraph C*-algebras which were introduced by Trieb, Weber and Zenner in [85]. Hypergraph C*-algebras are a direct generalization of graph C*-algebras which go back to [55, 94] and have been extensively studied since, see e.g. [71]. Graph C*-algebras are themselves a generalization of Cuntz- and Cuntz-Krieger algebras [24, 25].

These classes of C*-algebras have been important in the development of the classification program for C*-algebras. Indeed, classification results for Cuntz-Krieger/graph C*-algebras were often developed in parallel to or predate shortly more general results for more general C*-algebras.

A first classification of simple Cuntz-Krieger algebras was given by Rørdam [76] in terms of K-theory. K-theory associates to every C*-algebra A two groups $K_0(A)$ and $K_1(A)$ that encode information about projections and unitaries in A and $M_n(A)$ for $n \geq 2$. Rørdam's classification result was shortly thereafter confirmed as a special case of Phillips classification theorem for Kirchberg algebras [68]. For Cuntz-Krieger algebras with finitely many ideals a classification can still be obtained by passing from K-theory to so-called filtered K-theory. In this setting a classification (up to stable isomorphism) was obtained by Restorff [73] and a corresponding theory for arbitrary C*-algebras was developed in [63]. The ultimate classification of graph C*-algebras was obtained by Eilers, Restorff, Ruiz and Sørensen in more geometric terms [32, 33]: Their theorem allows to determine whether two graph C*-algebras $C^*(\Gamma_1)$ and $C^*(\Gamma_2)$ are (stably) isomorphic by checking if Γ_1 and Γ_2 are related by a sequence of explicit "moves" which transform a given graph into another one.

Hypergraph C*-algebras are a major generalization of graph C*-algebras with very different structural properties. Since their introduction by Trieb, Weber and Zenner in [85] an important aim is to understand where hypergraph C*-algebras differ from graph C*-algebras and to which extent results for graph C*-algebras carry over to hypergraph C*-algebras. In this chapter, we recall the definition and first properties of hypergraph C*-algebras from the fundamental paper [85]. We also discuss the author's results on nuclearity of hypergraph C*-algebras from [80] in more detail, and explain how they motivate the study of bipartite graph C*-algebras.

5.1. Definition and properties of hypergraph C*-algebras

Hypergraph C*-algebras were first introduced by Trieb, Weber and Zenner in [85]. The definition was slightly modified by the author in [80] to make sense of C*-algebras associated to undirected hypergraphs.

Definition 5.1.1. We have the following definitions:

1. A *hypergraph* $\mathrm{H}\Gamma$ is a tuple (E^0, E^1, r, s) where E^0 is a finite set of vertices, E^1 is a finite set of edges, and $r : E^1 \to \mathcal{P}(E^0)$ and $s : E^1 \to \mathcal{P}(E^0) \setminus \{\emptyset\}$ are maps called the range and source map, respectively. We say that $\mathrm{H}\Gamma$ is *undirected* if $r(e) = \emptyset$ holds for all $e \in E^1$.

2. The *hypergraph C*-algebra* $C^*(\mathrm{H}\Gamma)$ associated to a hypergraph $\mathrm{H}\Gamma$ is the universal C*-algebra generated by pairwise orthogonal projections $(p_v)_{v\in E^0}$ and partial isometries $(s_e)_{e\in E^1}$ satisfying the relations

$$s_e^* s_f = \begin{cases} \delta_{ef} \sum_{v\in r(e)} p_v, & r(e) \neq \emptyset, \\ \delta_{ef} s_e, & \text{otherwise,} \end{cases} \tag{HR1}$$

$$s_e s_e^* \leq \sum_{v\in s(e)} p_v, \tag{HR2}$$

$$p_v \leq \sum_{e\in E^1 : v\in s(e)} s_e s_e^*, \quad \text{if } v \text{ is not a sink,} \tag{HR3}$$

for all $e, f \in E^1$ and $v \in E^0$. In (HR3) a vertex v is called a sink if it is not contained in the source of any edge, i.e. if $v \notin s(e)$ holds for all $e \in E^1$.

If $\mathrm{H}\Gamma$ satisfies $|s(e)| = |r(e)| = 1$ for all $e \in E^1$, then $\mathrm{H}\Gamma$ is naturally identified with a directed graph Γ, and in this case $C^*(\mathrm{H}\Gamma)$ is isomorphic to the graph C*-algebra $C^*(\Gamma)$.

Proposition 5.1.2 ([85, Proposition 3.4])**.** *Let $\mathrm{H}\Gamma = (E^0, E^1, r, s)$ be a hypergraph such that $|s(e)| = |r(e)| = 1$ holds for all $e \in E^1$. Then a directed graph $\Gamma = (\tilde{E}^0, \tilde{E}^1, \tilde{r}, \tilde{s})$ is given by $\tilde{E}^0 := E^0$, $\tilde{E}^1 := E^1$, $\tilde{r}(e) := v$ and $\tilde{s}(e) := w$ for all $e \in E^1$ with $r(e) = \{v\}$ and $s(e) = \{w\}$. In this case, we have*

$$C^*(\mathrm{H}\Gamma) \cong C^*(\Gamma),$$

where $C^*(\Gamma)$ is the graph C-algebra associated to Γ.*

However, several nice properties of graph C*-algebras fail to hold for general hypergraph C*-algebras. Most notably, Trieb, Weber and Zenner proved in [85] that hypergraph C*-algebras fail to satisfy the Gauge Uniqueness Theorem and they generally need not be nuclear.

The Gauge Uniqueness Theorem is a fundamental property of graph C*-algebras: Assume that $C^*(\Gamma)$ is a graph C*-algebra and $\varphi : C^*(\Gamma) \to B$ is some $*$-homomorphism into a C*-algebra B. There is a natural action, called the *gauge action*, of the group S^1 on $C^*(\Gamma)$ given by $\gamma_z(p_v) = p_v$ and $\gamma_z(s_e) = z s_e$ for all $v \in E^0$, $e \in E^1$ and $z \in S^1$. The Gauge Uniqueness Theorem states that φ is injective as soon as there exists an action $\beta : S^1 \to \mathrm{Aut}(B)$ such that $\beta_z \circ \varphi = \varphi \circ \gamma_z$ holds for all $z \in S^1$. There is no problem to define the gauge action γ on a hypergraph C*-algebra $C^*(\mathrm{H}\Gamma)$ in the same way as for graph C*-algebras, but the Gauge Uniqueness Theorem fails to hold for arbitrary hypergraph C*-algebras.

Proposition 5.1.3 ([85, Proposition 4.2, Example 5.3]). *Let $n \geq 2$ and consider the hypergraph* $\mathrm{H}\Gamma$ *with* $E^0 = \{v_1, v_2, \dots, v_n\}$*,* $E^1 = \{e\}$*,* $s(e) = r(e) = \{v_1, v_2, \dots, v_n\}$*. Then*

$$C^*(\mathrm{H}\Gamma) \cong C(S^1) *_{\mathbb{C}} \mathbb{C}^n,$$

and $C^*(\mathrm{H}\Gamma)$ *is not nuclear. Further, there is a non-injective $*$-homomorphism*

$$\varphi : C^*(\mathrm{H}\Gamma) \to C(S^1) \otimes \mathbb{C}^n$$

and an action $\beta : S^1 \to \mathrm{Aut}(C(S^1) \otimes \mathbb{C}^n)$ *such that* $\beta_z \circ \varphi = \varphi \circ \gamma_z$ *holds for all* $z \in S^1$*. Here, the gauge action* $\gamma : S^1 \to \mathrm{Aut}(C^*(\mathrm{H}\Gamma))$ *is determined by*

$$\gamma_z(p_v) = p_v, \quad \textit{and} \quad \gamma_z(s_e) = zs_e$$

for all $v \in E^0$*,* $e \in E^1$ *and* $z \in S^1$*.*

5.2. Nuclearity of hypergraph C*-algebras

As discussed above, there are non-nuclear hypergraph C*-algebras. A natural question is to characterize exactly which hypergraphs give rise to nuclear hypergraph C*-algebras. This problem was studied by the author in his Master's thesis and the ensuing joint paper with Moritz Weber [80]. In this section, we briefly recall the obtained results to explain the connection to bipartite graph C*-algebras. We will not go into the details of the arguments, but remain on a rather verbose level.

As a first step, [80, Definition 3.1] introduces a number of so-called minor operations on hypergraphs which give rise to a partial order $\leq$ on hypergraphs, where $\mathrm{H}\Gamma \leq \mathrm{H}\Delta$ means that $\mathrm{H}\Gamma$ is obtained by applying a sequence of these operations to $\mathrm{H}\Delta$ successively. These operations resemble the moves on graph from Eilers, Restorff, Ruiz and Sørensen's classification of graph C*-algebras, but they serve a slightly different purpose: They are not supposed to preserve the C*-algebra up to stable isomorphism but they should make the C*-algebra "smaller" in a controlled way. In [80, Theorem 3.3], for each minor operation on the hypergraph level it is described exactly how the associated hypergraph C*-algebras are related. For instance if a vertex in the hypergraph is deleted then the C*-algebra of the resulting hypergraph is a quotient of a subalgebra of the original hypergraph C*-algebra.

The aim is to prove a forbidden minor characterization of nuclearity, i.e. we want to find a number of specific hypergraphs $\mathrm{H}\Gamma_i$ such that $C^*(\mathrm{H}\Gamma)$ is nuclear if, and only if, $\mathrm{H}\Gamma_i \leq \mathrm{H}\Gamma$ does not hold for any of the hypergraphs $\mathrm{H}\Gamma_i$. However, this is only obtained up to some extent. As one thing, it turns out to be necessary to "normalize" a given hypergraph $\mathrm{H}\Gamma$ in some way. For this [80, Algorithms 1 & 2] provide a method that transforms $\mathrm{H}\Gamma$ into another hypergraph $\mathrm{H}\Delta$ such that $C^*(\mathrm{H}\Gamma)$ is nuclear if and only if $C^*(\mathrm{H}\Delta)$ is nuclear.

It remains to determine whether $C^*(\mathrm{H}\Delta)$ is nuclear or not. There are two possiblities: If $\mathrm{H}\Gamma_i \leq \mathrm{H}\Delta$ holds for one of four specific hypergraphs $\mathrm{H}\Gamma_1, \mathrm{H}\Gamma_2, \mathrm{H}\Gamma_3$ and $\mathrm{H}\Gamma_4$, then

$C^*(\mathrm{H}\Delta)$ is not nuclear. Otherwise, we can not tell for sure whether $C^*(\mathrm{H}\Delta)$ is nuclear or not. However, we know that if $\mathrm{H}\Gamma_i \leq \mathrm{H}\Delta$ does not hold for any of the four specific hypergraphs, then $\mathrm{H}\Delta$ is an undirected hypergraph. Hence, if we knew which undirected hypergraph C*-algebras are nuclear, then we could determine whether $C^*(\mathrm{H}\Delta)$ is nuclear or not, and thus also whether $C^*(\mathrm{H}\Gamma)$ is nuclear or not. In this sense, the problem of characterizing nuclearity for arbitrary hypergraph C*-algebras is reduced to the same problem for undirected hypergraph C*-algebras.

5.3. Connection to bipartite graph C*-algebras

Finally, let us discuss the connection between bipartite graph C*-algebras and hypergraph C*-algebras. In fact, bipartite graph C*-algebras are exactly the hypergraph C*-algebras associated to undirected hypergraphs.

Proposition 5.3.1. *Let* $\mathrm{H}\Gamma = (E^0, E^1, s)$ *be an undirected hypergraph and let the bipartite graph* $G = (U, V, E)$ *be given by*

$$U = E^0, \qquad V = E^1, \qquad E = \{\{v, e\} \in E^0 \times E^1 \mid v \in s(e)\}.$$

Then $C^*(\mathrm{H}\Gamma) \cong C^*(G)$ *as* C^**-algebras.*

Proof. Let $(p_v)_{v\in E^0}$ and $(s_e)_{e\in E^1}$ be the generators of $C^*(\mathrm{H}\Gamma)$ and let $(\hat{p}_x)_{x\in U\cup V}$ be the generators of $C^*(G)$ as in Definition 4.2.1. Then we define a $*$-homomorphism $\varphi : C^*(G) \to C^*(\mathrm{H}\Gamma)$ by asking

$$\begin{aligned} \varphi(\hat{p}_x) &= p_x, && \text{if } x \in E^0, \\ \varphi(\hat{p}_x) &= s_x, && \text{if } x \in E^1. \end{aligned}$$

Crucially, the elements $s_x \in C^*(\mathrm{H}\Gamma)$ with $x \in E^1$ are projections because of the special case in (HR1) for edges with empty range. It is an easy exercise to check that the relations (GP1) and (GP2) are satisfied by the elements $\varphi(\hat{p}_x)$. Thus, the universal property of $C^*(G)$ yields that the map φ exists and is completely determined by the above prescriptions. Similarly, one defines the inverse map $\psi : C^*(\mathrm{H}\Gamma) \to C^*(G)$ as the unique $*$-homomorphism satisfying

$$\begin{aligned} \psi(p_v) &= \hat{p}_v, && \text{for all } v \in E^0, \\ \psi(s_e) &= \hat{p}_e, && \text{for all } e \in E^1. \end{aligned}$$

This homomorphism exists by the universal property of $C^*(G)$, and it is straightforward to check that ψ is the inverse of φ. Hence φ is an isomorphism, as desired. □

6. Classification of bipartite graph C*-algebras

Recall from Definition 2.4.4 that to every C*-algebra B is associated a topological space $\mathrm{Spec}(B)$ which contains as elements the equivalence classes of irreducible representations with respect to unitary equivalence. The goal of this chapter is to classify bipartite graph C*-algebras $C^*(G)$ up to isomorphism by properties of their spectrum. In fact, we do not consider the whole spectrum $\mathrm{Spec}(C^*(G))$ but only the subspace consisting of one- and two-dimensional irreducible representations. We denote this space by $\mathrm{Spec}_{\leq 2}(C^*(G))$, i.e. we set

$$\mathrm{Spec}_{\leq 2}(C^*(G)) := \{\pi \in \mathrm{Spec}(C^*(G)) \mid \pi \text{ is one- or two-dimensional}\}.$$

Let us briefly outline the approach for classifying bipartite graph C*-algebras. Our fundamental observation is that the elements of $\mathrm{Spec}_{\leq 2}(C^*(G))$ can be related to the vertices and edges in the graph G. Using this connection, we obtain from a homeomorphism $\Phi : \mathrm{Spec}_{\leq 2}(C^*(G)) \to \mathrm{Spec}_{\leq 2}(C^*(G'))$ a bijection $f : E \to E'$ between the edges of G and G'. In a second step, the map $f : E \to E'$ gives rise to a $*$-isomorphism $\varphi_f : C^*(G) \to C^*(G')$.

6.1. One- and two-dimensional irreducible representations

In this section, we analyze the one- and two-dimensional irreducible representations of bipartite graph C*-algebras $C^*(G)$. The spectrum of $C^*(K_{2,2})$ will serve as a basic building block to describe $\mathrm{Spec}_{\leq 2}(C^*(G))$ for arbitrary bipartite graphs G.

6.1.1. The complete graph $K_{2,2}$

As mentioned before, the C*-algebra $C^*(K_{2,2})$ of the complete bipartite graph $K_{2,2}$ is isomorphic to the universal C*-algebra $C^*(p, q)$ generated by two projections p and q which was described very explicitly by Pedersen (see Theorem 3.1.2). Let us state the isomorphism explicitly using the labels from Figure 6.1 for edges and vertices in $K_{2,2}$.

Proposition 6.1.1. *We have $C^*(K_{2,2}) \cong C^*(p, q)$. An isomorphism is given by the assignment*

$$\begin{aligned} p_{u_1} &\mapsto p, & p_{v_1} &\mapsto q \\ p_{u_2} &\mapsto 1 - p, & p_{v_2} &\mapsto 1 - q. \end{aligned}$$

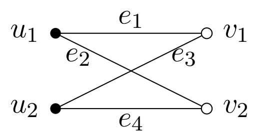


Figure 6.1.: The complete bipartite graph $K_{2,2}$

Figure 6.2.: The space X

Proof. Evidently, $\{p, 1-p\}$ and $\{q, 1-q\}$ are two partitions of unity. Since $K_{2,2}$ is complete, there are no particular orthogonality relations that the generators of $C^*(K_{2,2})$ must satisfy, i.e. (GP2) is trivial. Thus, the universal property of $C^*(K_{2,2})$ yields a $*$-homomorphism $\varphi : C^*(K_{2,2}) \to C^*(p,q)$ which extends the assignment from the statement of the proposition. Analogously, the universal property of $C^*(p,q)$ yields an inverse $*$-homomorphism $\psi : C^*(p,q) \to C^*(K_{2,2})$ with

$$\psi : p \mapsto p_{u_1},\ q \mapsto p_{v_1}.$$

□

Using Pedersen's description of $C^*(p,q)$ as subalgebra of $C([0,1], M_2)$, it is not difficult to describe the spectrum of $C^*(p,q)$. For that, let X be the quotient of the disjoint union

$$[0,1] \amalg [0,1] = \{t, t' : 0 \leq t \leq 1\}$$

over the equivalence relation that identifies t and t' for $t \in (0,1)$. The underlying set of this space has the form $\{a,b,c,d\} \cup (0,1)$ and can be sketched as in Figure 6.2. A neighborhood system of a is given by $\{\{a\} \cup (0, \frac{1}{n}) : n \in \mathbb{N}\}$, and there are analogous neighborhood systems for b, c and d. Note that the space X is T_0 but not Hausdorff.

Corollary 6.1.2. *We have*

$$\operatorname{Spec}_{\leq 2}(C^*(K_{2,2})) = \operatorname{Spec}(C^*(K_{2,2})) \cong X.$$

The points $a, b, c, d \in X$ correspond to the one-dimensional representations of $C^*(K_{2,2})$, while the points in $(0,1) \subset X$ correspond to the two-dimensional irreducible representations.

Proof. Let $K_{2,2}$ be labeled as in Figure 6.1. In view of Pedersen's result (Theorem 3.1.2) and Proposition 6.1.1 we may assume that the generators p_x of $C^*(K_{2,2})$ are concretely given by the functions

$$p_{u_1} : [0,1] \ni t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad p_{v_1} : [0,1] \ni t \mapsto \begin{pmatrix} 1-t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & t \end{pmatrix},$$
$$p_{u_2} : [0,1] \ni t \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad p_{v_2} : [0,1] \ni t \mapsto \begin{pmatrix} t & -\sqrt{t(1-t)} \\ -\sqrt{t(1-t)} & 1-t \end{pmatrix}.$$

One easily checks that the two-dimensional irreducible representations of $C^*(K_{2,2})$ are exactly the following (up to unitary equivalence):

$$\pi_t : C^*(K_{2,2}) \to M_2,\ f \mapsto f(t), \qquad \text{for } t \in (0,1),$$

while the one-dimensional irreducible representations are exactly (up to unitary equivalence):

$$\pi_a : C^*(K_{2,2}) \to \mathbb{C},\ f \mapsto f(0)_{11}, \qquad \pi_c : C^*(K_{2,2}) \to \mathbb{C},\ f \mapsto f(1)_{11},$$
$$\pi_b : C^*(K_{2,2}) \to \mathbb{C},\ f \mapsto f(0)_{22}, \qquad \pi_d : C^*(K_{2,2}) \to \mathbb{C},\ f \mapsto f(1)_{22}.$$

Together, these are all irreducible representations of $C^*(K_{2,2})$ up to unitary equivalence. By looking at the hull-kernel topology of the primitive ideal space

$$\mathrm{Prim}(C^*(K_{2,2})) = \{\pi_t^{-1}(0) : t \in (0,1)\} \cup \{\pi_a^{-1}(0), \pi_b^{-1}(0), \pi_c^{-1}(0), \pi_d^{-1}(0)\}$$

one confirms that the map

$$\varphi : \begin{cases} \mathrm{Spec}(C^*(K_{2,2})) \to X, & \\ \qquad\qquad \pi_t \mapsto t, & t \in (0,1), \\ \qquad\qquad \pi_a \mapsto a, & \\ \qquad\qquad \pi_b \mapsto b, & \\ \qquad\qquad \pi_c \mapsto c, & \\ \qquad\qquad \pi_d \mapsto d, & \end{cases}$$

is a homeomorphism of topological space. This concludes the proof. □

6.1.2. General situation

In what follows we investigate the connection between an arbitrary bipartite graph G and the topological space $\mathrm{Spec}_{\leq 2}(C^*(G))$. It turns out that the one-dimensional irreducible representations correspond to the edges of the graph G, while the two-dimensional ones correspond to subgraphs that are isomorphic to $K_{2,2}$. This allows for a complete description of $\mathrm{Spec}_{\leq 2}(C^*(G))$ from the combinatorial structure of G.

Throughout this section let $G = (U, V, E)$ be a bipartite graph and let

$$\mathcal{G} = \{H \subset G \mid H \cong K_{2,2}\}$$

be the collection of all subgraphs of G that are isomorphic to $K_{2,2}$.

Lemma 6.1.3. *The following statements hold:*

(a) Let π be a one-dimensional irreducible representation of $C^*(G)$. Then there is an edge $\{u_0, v_0\} \in E$ such that for all vertices $x \in U \cup V$ one has

$$\pi(p_x) = \begin{cases} 1, & \text{if } x \in \{u_0, v_0\}, \\ 0, & \text{otherwise.} \end{cases} \tag{6.1}$$

Moreover, for every edge $e \in E$ there is exactly one such irreducible representation, which we denote by π_e.

(b) Let σ be a two-dimensional irreducible representation of $C^*(G)$. Then there are vertices $u_1, u_2, v_1, v_2 \in U \cup V$ with $G(u_1, u_2, v_1, v_2) \cong K_{2,2}$ such that up to a unitary transformation one has

$$\begin{aligned} \sigma(p_{u_1}) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma(p_{v_1}) = \begin{pmatrix} 1-t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & t \end{pmatrix}, \\ \sigma(p_{u_2}) &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma(p_{v_2}) = \begin{pmatrix} t & -\sqrt{t(1-t)} \\ -\sqrt{t(1-t)} & 1-t \end{pmatrix}, \\ \sigma(p_x) &= 0 \qquad \text{for all } x \in (U \cup V) \setminus \{u_1, u_2, v_1, v_2\} \end{aligned} \tag{6.2}$$

for some $t \in (0,1)$.

Conversely, for every such vertices and some $t \in (0,1)$ there exists exactly one such irreducible representation, which we denote by $\sigma_{H,t}$ for $H = G(u_1, u_2, v_1, v_2)$.

In particular, one has the identity of sets

$$\operatorname{Spec}_{\leq 2}(C^*(G)) = \{\pi_e \mid e \in E\} \cup \{\sigma_{H,t} \mid H \in \mathcal{G}, t \in (0,1)\}.$$

Proof. Ad (a): Clearly,

$$1 = \pi(1) = \sigma\left(\sum_{u \in U} p_u\right) = \pi\left(\sum_{v \in V} p_v\right),$$

and $\pi(p_u), \pi(p_v) \in \{0,1\}$ for all $u \in U, v \in V$. Hence, there exist $u_0 \in U$ and $v_0 \in V$ with $1 = \pi(p_{u_0}) = \pi(p_{v_0})$. Assume $u_0 \not\sim v_0$. Using the relations (GP1) and (GP2) one obtains

$$\begin{aligned} 1 = \pi(p_{u_0}) = \pi\left(p_{u_0}\left(\sum_{v \in V} p_v\right)\right) &= \pi\left(p_{u_0}\left(\sum_{v \in \mathcal{N}(u_0)} p_v\right)\right) \\ &= \pi(p_{u_0}) \sum_{v \in \mathcal{N}(u_0)} \pi(p_v) = 0. \end{aligned}$$

This is a contradiction. Thus, we must have $u_0 \sim v_0$.

For the last statement, assume that vertices $u_0 \sim v_0$ are given. Then the universal property of $C^*(G)$ yields a (unique) $*$-homomorphism $\pi : C^*(G) \to \mathbb{C}$ with (6.1). Indeed, it is not hard to check that the elements $\pi(p_u)$ and $\pi(p_v)$ satisfy the relations (GP1) and (GP2) required from $C^*(G)$. Evidently, π is irreducible.

Ad (b): As

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sum_{u \in U} \sigma(p_u) = \sum_{v \in V} \sigma(p_v)$$

and every $\sigma(p_x)$ for $x \in U \sqcup V$ is a projection, there are vertices u_1, u_2, v_1 and v_2 with

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma(p_{u_1}) + \sigma(p_{u_2}) = \sigma(p_{v_1}) + \sigma(p_{v_2})$$

and

$$\sigma(p_x) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

for all $x \in (U \sqcup V) \setminus \{u_1, u_2, v_1, v_2\}$. For a contradiction assume

$$\sigma(p_{u_1}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma(p_{u_2}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, $\mathrm{im}(\sigma)$ is the closed span of $\sigma(p_{v_1})$ and $\sigma(p_{v_2})$ which are two orthogonal projections that sum up to the unit. This would mean, evidently, that σ is not irreducible – contradicting the assumption. Thus, all four matrices $\sigma(p_{u_1}), \sigma(p_{u_2}), \sigma(p_{v_1})$ and $\sigma(p_{v_2})$ are projections onto one-dimensional subspaces of $\mathbb{C}^2$. By applying a unitary transformation if necessary, we can assume $\sigma(p_{u_1}) = E_{11}$ and $\sigma_{p_{u_2}} = E_{22}$. Then, $\sigma(p_{v_1})$ is the projection onto some non-vanishing vector $ae_1 + be_2 \in \mathbb{C}^2$, and by applying a diagonal unitary transformation we can assume $a, b \in [0, \infty)$. After normalizing this vector, we have that $\sigma(p_{v_1})$ is the projection onto $te_1 + (1-t)e_2$ for some $t \in [0,1]$ with $t^2 + (1-t)^2 = 1$. Consequently, we get

$$\sigma(p_{v_1}) = \begin{pmatrix} 1-t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & t \end{pmatrix} \quad \text{and} \quad \sigma(p_{v_2}) = \begin{pmatrix} t & -\sqrt{t(1-t)} \\ -\sqrt{t(1-t)} & 1-t \end{pmatrix}.$$

Further, one observes immediately $t \notin \{0,1\}$ since otherwise σ would not be irreducible. It follows, in particular, $\sigma(p_{u_i})\sigma(p_{v_j}) \neq 0$ for $i, j \leq 2$. Similarly as before, (GP1) and (GP2) then entail $u_i \sim v_j$ for all i, j which means that $G(u_1, u_2, v_1, v_2) \cong K_{2,2}$.

For the final statement, one readily checks that the elements $\sigma(p_u)$ and $\sigma(p_v)$ given by (6.2) satisfy the relations (GP1) and (GP2). Therefore, the universal property of $C^*(G)$ yields a (unique) $*$-homomorphism $\sigma : C^*(G) \to M_2$ satisfying (6.2). Evidently, $\mathrm{im}(\sigma) = M_2$, and therefore the representation σ is irreducible. $\square$

Example 6.1.4. Let us again take a look at the complete bipartite graph $K_{2,2}$ which was discussed in the previous section. There we saw that the spectrum of $C^*(K_{2,2})$ is isomorphic to the space $X = \{a, b, c, d\} \sqcup (0, 1)$, where the points a, b, c, d correspond to one-dimensional irreducible representations. A sketch of this space can be seen in Figure 6.2. The points a and b (c and d) have no disjoint open neighborhoods, while all other pairs of distinct points from $\{a, b, c, d\}$ have disjoint open neighborhoods. By the previous lemma, these points correspond to the four edges of $K_{2,2}$. In the proof of Corollary 6.1.2 the irreducible representations π_a, π_b, π_c and π_d corresponding to the points $a, b, c, d \in X$ were described explicitly. Let us now check which edges correspond to which points. In the terminology of Corollary 6.1.2, we have

$$\begin{aligned} C^*(K_{2,2}) &= \{f \in C([0,1], M_2) \mid f(0), f(1) \text{ are diagonal matrices}\}, \\ p_{u_1} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{t\in[0,1]}, \quad p_{v_1} = \begin{pmatrix} 1-t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & t \end{pmatrix}_{t\in[0,1]}, \\ p_{u_2} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_{t\in[0,1]}, \quad p_{v_2} = \begin{pmatrix} t & -\sqrt{t(1-t)} \\ -\sqrt{t(1-t)} & 1-t \end{pmatrix}_{t\in[0,1]}, \end{aligned}$$

and the representations π_a, π_b, π_c and π_d are given by

$$\begin{aligned} &\pi_a : C^*(K_{2,2}) \to \mathbb{C},\ f \mapsto f(0)_{11}, \qquad &&\pi_c : C^*(K_{2,2}) \to \mathbb{C},\ f \mapsto f(1)_{11}, \\ &\pi_b : C^*(K_{2,2}) \to \mathbb{C},\ f \mapsto f(0)_{22}, \qquad &&\pi_d : C^*(K_{2,2}) \to \mathbb{C},\ f \mapsto f(1)_{22}. \end{aligned}$$

It follows that $\pi_a(u_1) = \pi_a(v_1) = 1$ and $\pi_a(u_2) = \pi_a(v_2) = 0$. Using the edge labels from Figure 6.1 we have $\{u_1, v_1\} = e_1$, and we see $\pi_a = \pi_{e_1}$, where π_{e_1} is the one-dimensional irreducible representation corresponding to the edge e_1 from the previous lemma. Checking the other representations in the same way, one arrives at the following correspondences:

$$\begin{array}{ccccc} a & \leftrightarrow & \pi_a = \pi_{e_1} & \leftrightarrow & e_1 = \{u_1, v_1\}, \\ b & \leftrightarrow & \pi_b = \pi_{e_4} & \leftrightarrow & e_4 = \{u_2, v_2\}, \\ c & \leftrightarrow & \pi_c = \pi_{e_2} & \leftrightarrow & e_2 = \{u_1, v_2\}, \\ d & \leftrightarrow & \pi_d = \pi_{e_3} & \leftrightarrow & e_3 = \{u_2, v_1\}. \end{array}$$

Interestingly, the pairs $\{a, b\}$ and $\{c, d\}$ of points without disjoint open neighborhoods correspond to the pairs of edges $\{e_1, e_4\}$ and $\{e_2, e_3\}$ which are the only two pairs of non-adjacent edges in $K_{2,2}$. Thus, the spectrum X of $C^*(K_{2,2})$ recalls which edges of $K_{2,2}$ are adjacent and which are not. This is an important feature which we will use later on. Therefore, we summarize the results of this example in the following lemma.

Lemma 6.1.5. *Let the complete graph $K_{2,2}$ have edge labels as in Figure 6.1. In $\mathrm{Spec}_{\leq 2}(C^*(K_{2,2}))$ the one-dimensional irreducible representations $\pi_{e_1}, \pi_{e_2}, \pi_{e_3}$ and π_{e_4} from Lemma 6.1.3 have the following property: Whenever e_i and e_j are two distinct edges, then π_{e_i} and π_{e_j} have disjoint open neighborhoods if and only if e_i and e_j are adjacent.*

Proof. See Example 6.1.4. The proof follows from the fact that the points a, b, c and d in X correspond to the edges e_1, e_2, e_3 and e_4 in such a way that $\{a, b\}$ and $\{c, d\}$ correspond to non-adjacent edges, while all other pairs of distinct points from $\{a, b, c, d\}$ correspond to adjacent edges. $\square$

In Lemma 6.1.6 we obtained a description of $\mathrm{Spec}_{\leq 2}(C^*(G))$ as a set. In the next lemma, we describe its topology using the space $X = \{a, b, c, d\} \sqcup (0, 1)$ from Corollary 6.1.2 as a building block.

Lemma 6.1.6. *Recall that $\mathcal{G} = \{H \subset G \mid H \subset G, H \cong K_{2,2}\}$ is the collection of all subgraphs of G that are isomorphic to $K_{2,2}$, and recall from Lemma 6.1.3 that*

$$\mathrm{Spec}_{\leq 2}(C^*(G)) = \{\pi_e \mid e \in E\} \sqcup \{\sigma_{H,t} \mid H \in \mathcal{G}, 0 < t < 1\}.$$

For every $H \in \mathcal{G}$ the set

$$X_H := \{\pi_{e_{H,1}}, \pi_{e_{H,2}}, \pi_{e_{H,3}}, \pi_{e_{H,4}}\} \sqcup \{\sigma_{H,t} \mid 0 < t < 1\} \subset \mathrm{Spec}_{\leq 2}(C^*(G))$$

is a closed subset that is isomorphic to the space X from Corollary 6.1.2, where $e_{H,1}, e_{H,2}, e_{H,3}$ and $e_{H,4}$ are the edges of H. Further, there is a homeomorphism

$$\bigcup_{H \in \mathcal{G}} \{\sigma_{H,t} \mid 0 < t < 1\} \cong \coprod_{H \in \mathcal{G}} (0, 1),$$

which maps every set $\{\sigma_{H,t} \mid 0 < t < 1\}$ onto a different copy of $(0, 1)$.

Finally, the following is true:

(a) π_e is clopen, i.e. $\{\pi_e\}$ is closed and open, in $\mathrm{Spec}_{\leq 2}(C^*(G))$ if e is not contained in any $H \in \mathcal{G}$,

(b) if $e_1, e_2 \in E$ are contained in some $H \in \mathcal{G}$ then π_{e_1} and π_{e_2} have disjoint open neighborhoods in $\mathrm{Spec}_{\leq 2}(C^*(G))$ if and only if e_1 and e_2 are adjacent,

(c) the points $\sigma_{H,t}$ have an open neighborhood in $\mathrm{Spec}_{\leq 2}(C^*(G))$ that is homeomorphic to $(0, 1)$ for every $H \in \mathcal{G}$ and $0 < t < 1$.

Proof. Let $H \in \mathcal{G}$ be fixed. One checks that the set X_H consists of all 1- and 2-dimensional irreducible representations of $C^*(G)$ that vanish on the projections p_x for $x \notin H$. Let $I_H \subset C^*(G)$ be the (closed) ideal generated by these projections p_x with $x \notin H$. By Proposition 2.4.5, $X_H \subset \mathrm{Spec}(C^*(G))$ is closed, and the map

$$\mathrm{Spec}(C^*(G)/I_H) \to \{\pi \in \mathrm{Spec}(C^*(G)) \mid \pi(p_x) = 0 \text{ for all } x \notin H\}, \quad \pi \mapsto \pi \circ \iota$$

is a homeomorphism, where $\iota : C^*(G) \to C^*(G)/I_H$ is the canonical quotient map. Since the map also preserves the dimension of the representations, we get a homeomorphism

$$\mathrm{Spec}_{\leq 2}(C^*(G)/I_H) \to \{\pi \in \mathrm{Spec}_{\leq 2}(C^*(G)) \mid \pi(p_x) = 0 \text{ for all } x \notin H\} = X_H,$$

where $X_H \subset \mathrm{Spec}_{\leq 2}(C^*(G))$ is a closed subset. By Proposition 4.2.5 the algebra $C^*(G)/I_H$ is isomorphic to $C^*(H) \cong C^*(K_{2,2})$. For this algebra we computed

$$\mathrm{Spec}_{\leq 2}(C^*(K_{2,2})) \cong X$$

in Corollary 6.1.2.

To prove the second statement, let us note that the sets $\{\sigma_{H,t} \mid 0 < t < 1\}$ are pairwise disjoint for different $H \in \mathcal{G}$ and isomorphic to $(0,1)$ because of $X_H \cong X$. For

$$\bigcup_{H \in \mathcal{G}} \{\sigma_{H,t} \mid 0 < t < 1\} \cong \coprod_{H \in \mathcal{G}} (0,1),$$

it suffices to show that $\{\sigma_{H,t} \mid 0 < t < 1\}$ is closed in the subspace topology of $\bigcup_{H \in \mathcal{G}} \{\sigma_{H,t} \mid 0 < t < 1\}$. However, this follows immediately from the fact that $X_H \subset \mathrm{Spec}_{\leq 2}(C^*(G))$ is closed. Thus, the map

$$\bigcup_{H \in \mathcal{G}} \{\sigma_{H,t} \mid 0 < t < 1\} \to \coprod_{H \in \mathcal{G}} (0,1),\ \sigma_{H,t} \mapsto t_H$$

is a homeomorphism, where for every $t \in (0,1)$ we denote by t_H the H-labeled copy of t in the disjoint union $\coprod_{H \in \mathcal{G}} (0,1)$.

It remains to prove (a)–(c). For (a), let $e \in E$ be an edge that is not contained in any $H \in \mathcal{G}$, and let G' be the graph obtained from G by deleting the edge e. From Corollary 4.2.7 one gets

$$C^*(G) \cong C^*(G') \oplus \mathbb{C},$$

and π_e corresponds to the representation

$$C^*(G') \oplus \mathbb{C} \ni (x,y) \mapsto y \in \mathbb{C}.$$

It follows immediately that $\{\pi_e\} \subset \mathrm{Spec}_{\leq 2}(C^*(G))$ is clopen.

For (b) let $H = G(e_1, e_2, e_3, e_4) \in \mathcal{G}$. After identifying $C^*(H)$ with $C^*(G)/I_H$ the above yields a homeomorphism

$$\begin{gathered}\mathrm{Spec}_{\leq 2}(C^*(H)) \to X_H \subset \mathrm{Spec}_{\leq 2}(C^*(G)),\\ \pi \mapsto \pi \circ \iota,\end{gathered}$$

where the quotient map $\iota : C^*(G) \to C^*(H)$ is given by

$$\iota(p_x) = \begin{cases} \hat{p}_x & \text{if } x \in H, \\ 0 & \text{if } x \notin H. \end{cases}$$

To avoid confusion we denote the generators of $C^*(H)$ by $\hat{p}_x$ for $x \in H$. Thus, for the edges e_i the homeomorphism maps

$$\hat{\pi}_{e_i} \mapsto \pi_{e_i},$$

where $\hat{\pi}_{e_i}$ is the one-dimensional irreducible representation of $C^*(H)$ corresponding to the edge e_i according to Lemma 6.1.3. As $H \cong K_{2,2}$, Lemma 6.1.5 tells us that e_1 and e_2 are adjacent in H if and only if $\hat{\pi}_{e_1}$ and $\hat{\pi}_{e_2}$ have disjoint open neighborhoods in $\mathrm{Spec}_{\leq 2}(C^*(H))$. It follows that π_{e_1} and π_{e_2} have disjoint open neighborhoods in $\mathrm{Spec}_{\leq 2}(C^*(G))$ if and only if e_1 and e_2 are adjacent in H. Evidently, this is equivalent to the fact that e_1 and e_2 are adjacent in G, which concludes the proof of (b).

Finally, for (c) let $H \in \mathcal{G}$ and $0 < t < 1$. It is not hard to check that $\{\sigma_{H,t} \mid 0 < t < 1\}$ is an open neighborhood of $\sigma_{H,t}$ in $\mathrm{Spec}_{\leq 2}(C^*(G))$ which is homeomorphic to $(0, 1)$. □

In the next lemma we obtain a bijection between the edges of two graphs G and G' starting from a homeomorphism between $\mathrm{Spec}_{\leq 2}(C^*(G))$ and $\mathrm{Spec}_{\leq 2}(C^*(G'))$. This will be used in the next section to obtain a $*$-isomorphism between $C^*(G)$ and $C^*(G')$.

Lemma 6.1.7. *Let G and G' be two bipartite graphs, and assume that*

$$\Phi : \mathrm{Spec}_{\leq 2}(C^*(G)) \to \mathrm{Spec}_{\leq 2}(C^*(G'))$$

is a homeomorphism. There is a unique bijection $f : E \to E'$ given by $\Phi(\pi_e) = \pi_{f(e)}$ for all edges $e \in E$. This map satisfies for all edges $e_1, e_2, e_3, e_4 \in E$

$$G(e_1, e_2, e_3, e_4) \cong K_{2,2} \quad \Leftrightarrow \quad G'(f(e_1), f(e_2), f(e_3), f(e_4)) \cong K_{2,2}. \tag{6.3}$$

Further, whenever $G(e_1, e_2, e_3, e_4) \cong K_{2,2}$, then one has for distinct $i, j \leq 4$

$$e_i \text{ and } e_j \text{ are adjacent} \quad \Leftrightarrow \quad f(e_i) \text{ and } f(e_j) \text{ are adjacent.} \tag{6.4}$$

Proof. Due to Lemma 6.1.6(a)–(c) the representations π_e for $e \in E$ and $\sigma_{H,t}$ for $H \in \mathcal{G}$ and $0 < t < 1$ can be distinguished by their topological properties. Therefore, the map Φ satisfies

$$\Phi\left(\{\pi_e \mid e \in E\}\right) = \{\pi_{e'} \mid e' \in E'\},$$
$$\Phi\left(\{\sigma_{H,t} \mid H \in \mathcal{G}, 0 < t < 1\}\right) = \{\sigma_{H',t} \mid H' \in \mathcal{G}', 0 < t < 1\}.$$

It follows that the map f is a well-defined bijection from E to E'. It remains to prove that f satisfies (6.3) and (6.4).

For that, let $X_H \subset \mathrm{Spec}_{\leq 2}(C^*(G))$ be the closed subset corresponding to a subgraph $H \in \mathcal{G}$ as in Lemma 6.1.6. We claim that for every $H \in \mathcal{G}$ there is an $H' \in \mathcal{G}'$ such that $\Phi(X_H) = X_{H'}$, and conversely for every $H' \in \mathcal{G}$ there is an $H \in \mathcal{G}$ with this property.

Indeed, X_H is the closure of $\{\sigma_{H,t} \mid 0 < t < 1\}$ and by the previous lemma we have

$$\bigcup_{H \in \mathcal{G}} \{\sigma_{H,t} \mid 0 < t < 1\} \cong \coprod_{H \in \mathcal{G}} (0, 1),$$
$$\bigcup_{H' \in \mathcal{G}'} \{\sigma_{H',t} \mid 0 < t < 1\} \cong \coprod_{H' \in \mathcal{G}'} (0, 1),$$

where the homeomorphism maps different sets $\{\sigma_{H,t} \mid 0 < t < 1\}$ for $H \in \mathcal{G}$ (or $\{\sigma_{H',t} \mid 0 < t < 1\}$ for $H' \in \mathcal{G}'$, resp.) onto different copies of $(0, 1)$. As a homeomorphism

Φ preserves connected components, and it follows that for every $H \in \mathcal{G}$ there exists a unique $H' \in \mathcal{G}'$ such that $\Phi(\{\sigma_{H,t} \mid 0 < t < 1\}) = \{\sigma_{H',t} \mid 0 < t < 1\}$. By taking the closure we obtain the claim. The same argument applied to Φ^{-1} shows that for every $H' \in \mathcal{G}'$ there is a unique $H \in \mathcal{G}$ such that $\Phi(X_H) = X_{H'}$.

Now, assume that $H = G(e_1, e_2, e_3, e_4) \in \mathcal{G}$. Then one has

$$\Phi(X_H) = X_{H'}$$

for some $H' = G'(e_1', e_2', e_3', e_4') \in \mathcal{G}'$. It follows

$$\begin{aligned}\Phi&(\{\pi_{e_1}, \pi_{e_2}, \pi_{e_3}, \pi_{e_4}\} \sqcup \{\sigma_{H,t} \mid 0 < t < 1\}) \\ &= \{\pi_{e_1'}, \pi_{e_2'}, \pi_{e_3'}, \pi_{e_4'}\} \sqcup \{\sigma_{H',t} \mid 0 < t < 1\}\end{aligned}$$

and thus

$$\{f(e_1), f(e_2), f(e_3), f(e_4)\} = \{e_1', e_2', e_3', e_4'\}.$$

This proves the forward implication in (6.3). The converse implication is proved in the same way using that for every $H' \in \mathcal{G}'$ there is a unique $H \in \mathcal{G}$ such that $\Phi(X_H) = X_{H'}$.

Finally, let $e_1, e_2 \in E$ be two distinct edges in some $H \in \mathcal{G}$. Then e_1 and e_2 are adjacent if and only if π_{e_1} and π_{e_2} have disjoint open neighborhoods in $\mathrm{Spec}_{\leq 2}(C^*(G))$ by Lemma 6.1.6(b). Similarly, $f(e_1)$ and $f(e_2)$ are adjacent if and only if $\pi_{f(e_1)}$ and $\pi_{f(e_2)}$ have disjoint open neighborhoods in $\mathrm{Spec}_{\leq 2}(C^*(G'))$. Since Φ is a homeomorphism we have that $\pi_{f(e_1)} = \Phi(\pi_{e_1})$ and $\pi_{f(e_2)} = \Phi(\pi_{e_2})$ have disjoint open neighborhoods in $\mathrm{Spec}_{\leq 2}(C^*(G'))$ if and only if π_{e_1} and π_{e_2} have disjoint open neighborhoods in $\mathrm{Spec}_{\leq 2}(C^*(G))$. This proves (6.4). □

6.2. Classification of bipartite graph C*-algebras

Finally, everything is ready to show that $\mathrm{Spec}_{\leq 2}(C^*(G))$ determines $C^*(G)$ up to isomorphism. Throughout this section let $G = (U, V, E)$ and $G' = (U', V', E')$ be two fixed bipartite graphs together with a homeomorphism

$$\Phi : \mathrm{Spec}_{\leq 2}(C^*(G)) \to \mathrm{Spec}_{\leq 2}(C^*(G')).$$

As seen in Lemma 6.1.7, the homeomorphism Φ yields a bijection $f : E \to E'$ between the respective edge sets of G and G' such that for any edges $e_1, e_2, e_3, e_4 \in E$ we have

$$G(e_1, e_2, e_3, e_4) \cong K_{2,2} \quad \Leftrightarrow \quad G'(f(e_1), f(e_2), f(e_3), f(e_4)) \cong K_{2,2}. \tag{6.3}$$

Additionally, if $G(e_1, e_2, e_3, e_4) \cong K_{2,2}$, then for $i, j \leq 4$ one has

$$e_i \text{ and } e_j \text{ are adjacent} \quad \Leftrightarrow \quad f(e_i) \text{ and } f(e_j) \text{ are adjacent.} \tag{6.4}$$

In this section, we prove that f gives rise to a $*$-isomorphism $\varphi : C^*(G) \to C^*(G')$. First, we show that any map $f : E \to E'$ satisfying (6.3) and (6.4) induces a $*$-homomorphism $\varphi_f : C^*(G) \to C^*(G')$.

Proposition 6.2.1. *Let G, G' be two bipartite graphs and let $f : E \to E'$ be a bijection between their edges which satisfies* (6.3) *and* (6.4). *Then there is a (unique) $*$-homomorphism $\varphi_f : C^*(G) \to C^*(G')$ such that for every vertex $x \in U \cup V$ of G one has*

$$\varphi_f(p_x) = \sum_{\{u',v'\} \in I(x)} \hat{p}_{u'}\hat{p}_{v'} =: P_x \in C^*(G'),$$

where

$$I(x) := \{f(e) \in E' \mid x \in e\}$$

and $\{u', v'\} \in E'$ implicitly requires that $u' \in U'$ and $v' \in V'$. Further, we denote the generators of $C^*(G')$ by $\hat{p}_y$ with $y \in U' \cup V'$ to avoid confusion with the generators p_x with $x \in U \cup V$ of $C^*(G)$.

To prove this, we need to show that the P_x are projections which satisfy the relations (GP1) and (GP2) required for the generators of $C^*(G)$, i.e.

- every P_x with $x \in U \cup V$ is a projection in $C^*(G')$,
- it is $\sum_{u \in U} P_u = 1 = \sum_{v \in V} P_v$, and
- we have $P_x \perp P_y$ unless $x \sim y$.

We will prove these properties in the next lemmas separately. Throughout, G, G' and f are fixed as in the statement of Proposition 6.2.1. Further, $x \in U \cup V$ is a fixed vertex in G and $I(x) = \{f(e) \mid x \in e\}$ is the set from Proposition 6.2.1. Let us start with a technical lemma.

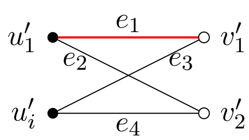
Lemma 6.2.2. *Let $v_1', v_2' \in V'$ be two distinct vertices of G' with common neighbors $u_1', \ldots, u_n' \in U'$. Assume*

$$\{u_1', v_1'\} \in I(x), \quad \textit{and} \quad \{u_1', v_2'\} \notin I(x).$$

Then for all $i \leq n$ we have

$$\{u_i', v_1'\} \in I(x), \quad \textit{and} \quad \{u_i', v_2'\} \notin I(x).$$

Proof. If $n \leq 1$ there is nothing to show. So assume $n > 1$ and let $i > 1$. The following is a subgraph of G'



where the edge e_1 (highlighted in red) is in $I(x)$. This means that $x \in f^{-1}(e_1)$. Since f has property (6.3) we know

$$G\left(f^{-1}(e_1), f^{-1}(e_2), f^{-1}(e_3), f^{-1}(e_4)\right) \cong K_{2,2}.$$

Hence, either e_2 or e_3 is in $I(x)$ while $e_4 = \{u_i', v_2'\} \notin I(x)$. By assumption, however, we have $e_3 = \{u_1', v_2'\} \notin I(x)$. Therefore, it follows immediately $e_2 = \{u_i', v_1'\} \in I(x)$ and this concludes the proof. $\square$

Lemma 6.2.3. *The element $P_x = \sum_{\{u',v'\}\in I(x)} \hat{p}_{u'}\hat{p}_{v'} \in C^*(G')$ from Proposition 6.2.1 is a projection. Further, the sets $\{P_u \mid u \in U\}$ and $\{P_v \mid v \in V\}$ form a partition of unity in $C^*(G')$, respectively, i.e. they satisfy relation* (GP1).

Proof. Observe for any $x \in U \cup V$

$$\begin{aligned}
P_x P_x^* &= \left(\sum_{\{u',v'\}\in I(x)} \hat{p}_{u'}\hat{p}_{v'}\right)\left(\sum_{\{u'',v''\}\in I(x)} \hat{p}_{u''}\hat{p}_{v''}\right)^* \\
&= \left(\sum_{\{u',v'\}\in I(x)} \hat{p}_{u'}\hat{p}_{v'}\right)\left(\sum_{\{u'',v''\}\in I(x)} \hat{p}_{v''}\hat{p}_{u''}\right) \\
&= \sum_{\{u',v'\},\{u'',v''\}\in I(x)} \hat{p}_{u'}\hat{p}_{v'}\hat{p}_{v''}\hat{p}_{u''} \\
&= \sum_{\{u',v'\},\{u'',v'\}\in I(x)} \hat{p}_{u'}\hat{p}_{v'}\hat{p}_{u''},
\end{aligned}$$

where we use (GP1) for the generators of $C^*(G')$ in the last step. Let u' and $u'' \in U'$ be fixed. We claim

$$\sum_{v':\{u',v'\},\{u'',v'\}\in I(x)} \hat{p}_{u'}\hat{p}_{v'}\hat{p}_{u''} = \left(\sum_{v':\{u',v'\}\in I(x)} \hat{p}_{u'}\hat{p}_{v'}\right)\hat{p}_{u''}.$$

To prove the claim let us distinguish the following cases.

Case 1. Assume $u' = u''$. Then the claim is immediate, for $\{u', v'\} \in I(x)$ and $\{u'', v'\} \in I(x)$ are equivalent.

Case 2. Assume $u' \neq u''$. Further, suppose that there is a common neighbor v_0' of u' and u'' such that

$$\{u', v_0'\} \in I(x), \quad \text{and} \quad \{u'', v_0'\} \notin I(x).$$

Then, by Lemma 6.2.2 the same holds for any common neighbor of u' and u'' instead of v_0'. Hence, we have on the one hand

$$\sum_{v':\{u',v'\},\{u'',v'\}\in I(x)} \hat{p}_{u'}\hat{p}_{v'}\hat{p}_{u''} = \sum_{\emptyset} = 0,$$

and on the other hand

$$\begin{aligned}\left(\sum_{v':\{u',v'\}\in I(x)} \hat{p}_{u'}\hat{p}_{v'}\right)\hat{p}_{u''} &= \hat{p}_{u'}\left(\sum_{v'\in\mathcal{N}(u')\cap\mathcal{N}(u'')} \hat{p}_{v'}\right)\hat{p}_{u''}\\ &= \hat{p}_{u'}\hat{p}_{u''}\\ &= 0,\end{aligned}$$

where we use in the first step that all common neighbors of u' and u'' are in $I(x)$ and products over non-neighbors vanish by (GP2), and in the last step relation (GP1) for the generators of $C^*(G')$.

Case 3. Finally, assume that neither of the previous cases applies. Then $u' \neq u''$ and for any common neighbor v' of u' and u'' we have

$$\{u',v'\}\in I(x) \implies \{u'',v'\}\in I(x).$$

Using the relations (GP2) for the generators of $C^*(G')$ one obtains

$$\begin{aligned}\sum_{v':\{u',v'\},\{u'',v'\}\in I(x)} \hat{p}_{u'}\hat{p}_{v'}\hat{p}_{u''} &= \sum_{v':\{u',v'\}\in I(x),\{u'',v'\}\in E'} \hat{p}_{u'}\hat{p}_{v'}\hat{p}_{u''}\\ &= \left(\sum_{v':\{u',v'\}\in I(x)} \hat{p}_{u'}\hat{p}_{v'}\right)\hat{p}_{u''}.\end{aligned}$$

Altogether we conclude

$$\begin{aligned}P_xP_x^* &= \sum_{\{u',v'\},\{u'',v'\}\in I(x)} \hat{p}_{u'}\hat{p}_{v'}\hat{p}_{u''}\\ &= \sum_{u',u''\in U'}\left(\sum_{v':\{u',v'\},\{u'',v'\}\in I(x)} \hat{p}_{u'}\hat{p}_{v'}\hat{p}_{u''}\right)\\ &= \sum_{u',u''\in U'}\left(\left(\sum_{v':\{u',v'\}\in I(x)} \hat{p}_{u'}\hat{p}_{v'}\right)\hat{p}_{u''}\right)\\ &= \left(\sum_{\{u',v'\}\in I(x)} \hat{p}_{u'}\hat{p}_{v'}\right)\left(\sum_{u''\in U'} \hat{p}_{u''}\right)\\ &= P_x,\end{aligned}$$

where we use (GP1) in the last step. This shows that P_x is a projection in $C^*(G')$ since one obtains directly $P_x^* = (P_x P_x^*)^* = P_x P_x^* = P_x$. Finally, observe

$$\begin{aligned}\sum_{u \in U} P_u &= \sum_{u \in U} \left(\sum_{\{u',v'\} \in I(u)} \hat{p}_{u'} \hat{p}_{v'} \right) \\ &= \sum_{\{u',v'\} \in E'} \hat{p}_{u'} \hat{p}_{v'} \\ &= \left(\sum_{u' \in U'} \hat{p}_{u'} \right) \left(\sum_{v' \in V'} \hat{p}_{v'} \right) \\ &= 1 \cdot 1 = 1,\end{aligned}$$

where we use (GP2) in the second-last step and (GP1) in the last step. This shows that the set $\{P_u \mid u \in U\}$ is a partition of unity in $C^*(G')$. The argument works analogously for $\{P_v \mid v \in V\}$. □

Lemma 6.2.4. *Let* $P_x = \sum_{\{u',v'\} \in I(x)} \hat{p}_{u'} \hat{p}_{v'} \in C^*(G')$ *be the element from Proposition 6.2.1. For any* $u \in U$ *and* $v \in V$ *we have* $P_u P_v = 0$ *unless* $u \sim v$*, i.e. the relations* (GP2) *are satisfied.*

Proof. By the previous Lemma 6.2.3 the P_x are projections. Observe

$$\begin{aligned}P_u P_v &= P_u P_v^* \\ &= \left(\sum_{\{u',v'\} \in I(u)} \hat{p}_{u'} \hat{p}_{v'} \right) \left(\sum_{\{u'',v''\} \in I(v)} \hat{p}_{v''} \hat{p}_{u''} \right) \\ &= \sum_{\{u',v'\} \in I(u), \{u'',v''\} \in I(v)} \hat{p}_{u'} \hat{p}_{v'} \hat{p}_{v''} \hat{p}_{u''} \\ &= \sum_{\{u',v'\} \in I(u), \{u'',v'\} \in I(v)} \hat{p}_{u'} \hat{p}_{v'} \hat{p}_{u''},\end{aligned}$$

where we use (GP2) in the last step. Assume that there are some $\{u', v'\} \in I(u)$ and $\{u'', v'\} \in I(v)$ such that

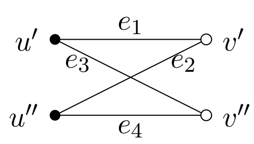
$$\hat{p}_{u'} \hat{p}_{v'} \hat{p}_{u''} \neq 0.$$

We distinguish two cases. First, if $u' = u''$, then we have

$$\{u', v'\} \in I(u) \cap I(v) = \{f(e) \mid e = \{u, v\} \in E\},$$

and therefore $u \sim v$.

Second, let us assume $u' \neq u''$. In view of Lemma 4.2.4 there must be some $v'' \neq v'$ such that the following is a subgraph of G'.



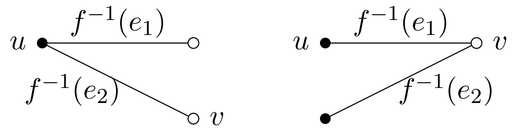


Figure 6.3.: The two possible forms of the subgraph $G(f^{-1}(e_1), f^{-1}(e_2))$

Using that f satisfies (6.3) it follows that the subgraph

$$G(f^{-1}(e_1), f^{-1}(e_2), f^{-1}(e_3), f^{-1}(e_4)) \subset G$$

is isomorphic to $K_{2,2}$. Because of (6.4) the edges $f^{-1}(e_1)$ and $f^{-1}(e_2)$ are adjacent. Using $u \in f^{-1}(e_1) \cap U$ and $v \in f^{-1}(e_2) \cap V$ one easily checks that the subgraph $G(f^{-1}(e_1), f^{-1}(e_2))$ can only take two different forms which are sketched in Figure 6.3. In both cases, one sees $u \sim v$. $\square$

Putting the previous lemmas together we obtain a proof of Proposition 6.2.1.

Proof of Proposition 6.2.1. The preceding lemmas show that the P_x satisfy the relations required for the generators of $C^*(G)$. Thus, by the universal property of $C^*(G)$ there is a unique $*$-homomorphism $\varphi_f : C^*(G) \to C^*(G')$ with $\varphi_f(p_x) = P_x$ for all $x \in U \cup V$. $\square$

Let us summarize what we have seen so far. Given a homeomorphism

$$\Phi : \mathrm{Spec}_{\leq 2}(C^*(G)) \to \mathrm{Spec}_{\leq 2}(C^*(G')),$$

we obtain from Lemma 6.1.7 a bijective map $f : E \to E'$ that satisfies the properties (6.3) and (6.4). Evidently, its inverse map $f^{-1} : E' \to E$ has the same properties. Thus, Proposition 6.2.1 yields two $*$-homomorphisms

$$\varphi_f : C^*(G) \to C^*(G') \quad \text{and} \quad \varphi_{f^{-1}} : C^*(G') \to C^*(G)$$

that are defined on the respective generators as follows:

$$\varphi_f : C^*(G) \ni p_x \mapsto P_x = \sum_{\{u',v'\} \in I(x)} \hat{p}_{u'} \hat{p}_{v'} \in C^*(G'), \qquad x \in U \cup V, \tag{6.5}$$

$$\varphi_{f^{-1}} : C^*(G') \ni \hat{p}_y \mapsto \hat{P}_y := \sum_{\{u,v\} \in J(y)} p_u p_v \in C^*(G), \qquad y \in U' \cup V', \tag{6.6}$$

where $I(x) = \{f(e) \in E' : x \in e\}$ and $J(x) = \{f^{-1}(e) \in E : y \in e\}$ for $x \in U \cup V$ and $y \in U' \cup V'$, respectively. In the above formulas we implicitly require $u \in U, v \in V$ and $u' \in U', v' \in V'$.

To prove $C^*(G) \cong C^*(G')$ it only remains to show that φ_f and $\varphi_{f^{-1}}$ are inverse to each other. In fact, it suffices to show $\varphi_{f^{-1}} \circ \varphi_f = \mathrm{id}_{C^*(G)}$ because the other statement $\varphi_f \circ \varphi_{f^{-1}} = \mathrm{id}_{C^*(G')}$ follows by replacing f with f^{-1}. We prove this in the following lemma.

Lemma 6.2.5. *One has* $\varphi_{f^{-1}} \circ \varphi_f = \mathrm{id}_{C^*(G)}$.

Proof. It suffices to show

$$\varphi_{f^{-1}}(\varphi_f(p_x)) = p_x,$$

for all $x \in U \cup V$. Indeed, we only prove this for $x \in U$ because the case $x \in V$ is analogous. Thus, let $u \in U$ be fixed. The definitions of $\varphi_{f^{-1}}$ and φ_f, respectively, yield

$$\begin{aligned}\varphi_{f^{-1}}(\varphi_f(p_u)) &= \varphi_{f^{-1}}\left(\sum_{\{u',v'\}\in I(u)} \hat{p}_{u'}\hat{p}_{v'}\right)\\ &= \sum_{\{u',v'\}\in I(u)} \left(\sum_{\{u_1,v_1\}\in J(u')} p_{u_1}p_{v_1}\right)\left(\sum_{\{u_2,v_2\}\in J(v')} p_{u_2}p_{v_2}\right).\end{aligned}$$

Using $\varphi_{f^{-1}}(\hat{p}_{u'}) = \left(\varphi_{f^{-1}}(\hat{p}_{u'})\right)^* = \sum_{\{u_1,v_1\}\in J(u')} p_{v_1}p_{u_1}$ together with (GP2) we can continue the computation as follows:

$$\begin{aligned}\varphi_{f^{-1}}(\varphi_f(p_u)) &= \sum_{\{u',v'\}\in I(u)} \left(\sum_{\{u_1,v_1\}\in J(u')} p_{v_1}p_{u_1}\right)\left(\sum_{\{u_2,v_2\}\in J(v')} p_{u_2}p_{v_2}\right)\\ &= \sum_{\{u',v'\}\in I(u)} \left(\sum_{\{u_1,v_1\}\in J(u'),\{u_1,v_2\}\in J(v')} p_{v_1}p_{u_1}p_{v_2}\right).\end{aligned}$$

We claim

$$\begin{aligned}&\left\{(v_1,u_1,v_2) \;\middle|\; \begin{matrix}\{u_1,v_1\}\in J(u')\\ \{u_1,v_2\}\in J(v')\end{matrix} \text{ for some } \{u',v'\}\in I(u) \text{ with } p_{v_1}p_{u_1}p_{v_2}\neq 0\right\}\\ &= \{(v_1,u,v_2) : V \ni v_1 \sim u \sim v_2 \in V \text{ with } p_{v_1}p_up_{v_2}\neq 0\},\end{aligned}$$

where u remains the fixed vertex from above. First, assume that (v_1, u, v_2) is in the set on the right-hand side, i.e.

$$V \ni v_1 \sim u \sim v_2 \in V \quad \text{with} \quad p_{v_1}p_up_{v_2} \neq 0.$$

There are two possibilities. If $v_1 = v_2$, then setting $\{u', v'\} := f(\{u, v_1\})$ one easily verifies $\{u, v_1\} \in J(u') \cap J(v')$ while $\{u', v'\} \in I(u)$. Thus, the triple (v_1, u, v_2) is in the set on the left-hand side.

Next, assume $v_1 \neq v_2$. As $p_{v_1}p_up_{v_2}$ is non-zero, Lemma 4.2.4 yields some vertex $u_2 \in U$ such that the graph from Figure 6.4 is a subgraph of G. As e_1 and e_2 are adjacent in $G(e_1, e_2, e_3, e_4)$, by Property (6.4) the same holds in $G'(f(e_1), f(e_2), f(e_3), f(e_4))$. Thus, the subgraph $G'(f(e_1), f(e_2))$ must take one of the two forms shown in Figure 6.5. Choosing the vertices $u' \in f(e_1)$ and $v' \in f(e_2)$ as in the figure one easily checks $\{u', v'\} \in I(u)$ as well as $e_1 = \{u, v_1\} \in J(u')$ and $e_2 = \{u, v_2\} \in J(v')$. Thus, the triple

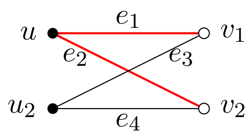


Figure 6.4.: The subgraph $G(u, v_1, v_2, u_2)$

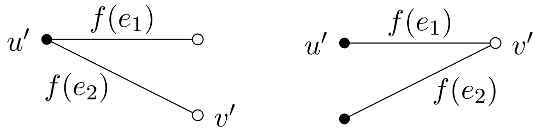


Figure 6.5.: The two possible forms of the subgraph $G'(f(e_1), f(e_2))$

(v_1, u, v_2) is in the set on the left-hand side, and this concludes the proof of the inclusion (lhs) $\supseteq$ (rhs).

For the other inclusion, assume that (v_1, u_1, v_2) is in the set on the left-hand side, i.e.

$$\begin{matrix} \{u_1, v_1\} \in J(u') \\ \{u_1, v_2\} \in J(v') \end{matrix} \quad \text{for some } \{u', v'\} \in I(u) \text{ with } p_{v_1} p_{u_1} p_{v_2} \neq 0.$$

It suffices to show $u_1 = u$ since $p_{v_1} p_{u_1} p_{v_2} \neq 0$ implies $v_1 \sim u_1 \sim v_2$ (using (GP2)) and thus the triple (v_1, u_1, v_2) is in the set on the right-hand side.

If $v_1 = v_2$, then we obtain immediately $f(\{u_1, v_1\}) = \{u', v'\} \in I(u)$ and this yields $u_1 = u$.

Next, assume $v_1 \neq v_2$. In view of Lemma 4.2.4 there exists some $u_2 \in U$ such that the graph from Figure 6.6 is a subgraph of G. where $e_1 \in J(u')$ ($\Leftrightarrow u' \in f(e_1)$), and $e_2 \in J(v')$ ($\Leftrightarrow v' \in f(e_2)$). Again by Property (6.4) the subgraph $G'(f(e_1), f(e_2))$ must take one of the two forms shown in Figure 6.5. The vertices u' and v' must be as depicted since $e_1 \in J(u')$ and $e_2 \in J(v')$. Now, one sees that either $f(e_1) = \{u', v'\}$ or $f(e_2) = \{u', v'\}$. Then the assumption $\{u', v'\} \in I(u)$ implies $f(e_1) \in I(u)$ or $f(e_2) \in I(u)$. In any event, it follows that $u_1 = u$.

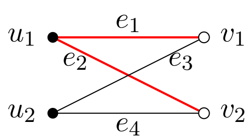


Figure 6.6.: The graph $G(u_1, v_1, v_2, u_2)$

Putting everything together we obtain

$$\begin{aligned}
\varphi_{f^{-1}}(\varphi_f(p_u)) &= \sum_{\{u',v'\}\in I(u)} \left(\sum_{\{u_1,v_1\}\in J(u'),\{u_1,v_2\}\in J(v')} p_{v_1}p_{u_1}p_{v_2} \right) \\
&= \sum_{V\ni v_1\sim u\sim v_2\in V} p_{v_1}p_u p_{v_2} \\
&= \sum_{v_1,v_2\in V} p_{v_1}p_u p_{v_2} \\
&= \left(\sum_{v_1\in V} p_{v_1}\right) p_u \left(\sum_{v_2\in V} p_{v_2}\right) \\
&= p_u,
\end{aligned}$$

where we use the claim for the step from the first to the second line. The last two equalities follows from (GP2) and (GP1), respectively. □

Finally, we can state the main result of this section. This theorem was mentioned in the introduction as Theorem C.

Theorem 6.2.6. *We have*

$$C^*(G) \cong C^*(G') \quad \Leftrightarrow \quad \mathrm{Spec}_{\leq 2}(C^*(G)) \cong \mathrm{Spec}_{\leq 2}(C^*(G')).$$

Proof. The implication from left to right is clear. So let

$$\Phi : \mathrm{Spec}_{\leq 2}(C^*(G)) \to \mathrm{Spec}_{\leq 2}(C^*(G'))$$

be a homeomorphism. One obtains an induced bijective map $f : E \to E'$ which satisfies (6.3) and (6.4) as discussed in Lemma 6.1.7. The same properties hold for the inverse map $f^{-1} : E' \to E$. By Proposition 6.2.1 we obtain the $*$-homomorphisms φ_f and $\varphi_{f^{-1}}$ described in (6.5) and (6.6). By Lemma 6.2.5, $\varphi_{f^{-1}}$ is a left-inverse of φ, and by replacing f with f^{-1} one gets that φ_f is a left-inverse of $\varphi_{f^{-1}}$. Consequently, φ_f is a $*$-isomorphism. □

7. Hypercube C*-algebras

In this chapter we study a particular class of bipartite graph C*-algebras, namely those that are associated to hypercubes. Hypercubes are the class of graphs that generalize the square and the cube to arbitrary dimensions. For every dimension $n \in \mathbb{N} \setminus \{0\}$ the n-dimensional hypercube Q_n is a bipartite graph with 2^n vertices. We will give a precise definition below, and, for the moment, content ourselves with a sketch of the three hypercubes Q_1, Q_2, Q_3 in Figure 7.1. Note that every hypercube is naturally interpreted as a bipartite graph.

There are several reasons to study hypercube C*-algebras: First, their C*-algebras provide a very satisfying generalization of Pedersen's theorem (Theorem 3.1.2). Second, they occur naturally in the study of hypergraph C*-algebras, and, finally, we can use the C*-algebra of the cube to solve a problem about magic isometries. This was discussed briefly in the introduction. At this point, we would like to elaborate on the significance of hypercube C*-algebras to hypergraph C*-algebras in more detail.

An open problem for hypergraph C*-algebras is the following:

Problem 7.0.1. For which hypergraphs $\mathrm{H}\Gamma$ is the hypergraph C*-algebra $C^*(\mathrm{H}\Gamma)$ nuclear?

This question was raised by Trieb, Weber and Zenner in [85] together with the observation that there are non-nuclear hypergraph C*-algebras. This is in stark contrast to graph C*-algebras, a class of C*-algebras that is generalized by hypergraph C*-algebras and which are always nuclear (see e.g. Raeburn's monograph [71]). Together with Moritz Weber the author proved in [80] a theorem with the following consequence: In order to tell which hypergraph C*-algebras are nuclear, it suffices to tell which bipartite graph C*-algebras are nuclear. Thus, we are led to the next problem:

Problem 7.0.2. For which bipartite graphs G is the bipartite graph C*-algebra $C^*(G)$ nuclear?

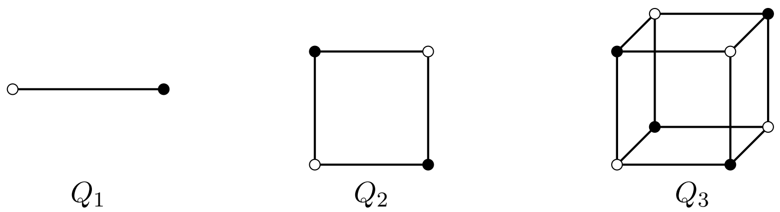


Figure 7.1.: The hypercubes Q_1, Q_2 and Q_3

Unfortunately, we are not able to answer this question in full generality. It is clear that $C^*(G)$ is not nuclear as soon as $K_{2,3} \subset G$ (see Corollary 4.2.6). However, it remains a problem whether $K_{2,3} \not\subset G$ entails that $C^*(G)$ is nuclear. Hypercube C*-algebras are a natural test case for this question. On the one hand, they satisfy $K_{2,3} \not\subset Q_n$ for all $n \in \mathbb{N}$. On the other hand, hypercubes are highly symmetric graphs which facilitates a detailed study of their C*-algebras. In fact, our generalization of Pedersen's theorem to hypercube C*-algebras will allow us to conclude that all hypercube C*-algebras are nuclear (Corollary 7.2.6).

Let us outline the rest of this chapter. After introducing notations about hypercubes, in **Section 7.2** we employ a combinatorial argument in the vein of Roch and Silbermann [75] to prove that every hypercube C*-algebra is subhomogeneous, and that every irreducible representation is, in our terminology, a rank-one representation. These are described in the next **Section 7.3** by certain edge-weightings of the hypercube. Using that description we are able to classify all irreducible representations of $C^*(Q_n)$ in **Section 7.5**. This allows us in **Section 7.6** to generalize Pedersen's theorem to hypercube C*-algebras. Finally, in **Section 7.7** we apply our results to the theory of magic isometries.

7.1. Hypercubes

In this section we give a precise definition of the hypercubes Q_n for $n \in \mathbb{N}$ using binary numbers. We also collect some properties of hypercubes that will be useful later.

The following notation will be crucial throughout this paper.

Notation 7.1.1. For every number $i \in \mathbb{N}$ and $n \in \mathbb{N} \setminus \{0\}$ such that $i < 2^n$ the n-digit binary representation of i is the sequence

$$i_{n-1} \dots i_1 i_0 \in \{0,1\}^n \quad \text{such that} \quad i = \sum_{k=0}^{n-1} i_k 2^k.$$

We call the number i_k the k-th binary digit of i and denote $[i]_k := i_k$. Further, we set

$$\mathrm{par}_k(i) = \sum_{\ell=0}^{k} i_\ell \mod 2.$$

Note that $\mathrm{par}_0(i)$ is the parity of the number i.

Further, for every $k < n$ let $i\#k \in \mathbb{N}$ be the number obtained by flipping the k-th binary digit of i, i.e.

$$i\#k = \begin{cases} i + 2^k & \text{if } i_k = 0, \\ i - 2^k & \text{if } i_k = 1. \end{cases}$$

Note that we will be using square brackets [ ] both to denote digits of a binary number $[i]_\ell$ and to indicate components of a vector $[\psi]_i$. It is always clear from the context which meaning is intended.

Definition 7.1.2. The n-dimensional hypercube $Q_n = (U_n, V_n, E_n)$ is the bipartite graph with vertex sets

$$U_n = \{i < 2^n \mid \mathrm{par}_{n-1}(i) = 0\}, \qquad V_n = \{j < 2^n \mid \mathrm{par}_{n-1}(j) = 1\}$$

and edge set

$$E_n = \{\{i, j\} \mid i \in U_n,\ j \in V_n,\ i\#k = j \text{ for some } k < n\}.$$

We denote the edge $\{i, j\} \in E_n$ with $i \in U_n$ and $j \in V_n$ also by $ij \in E_n$.

Example 7.1.3. Let us take a second look at the three hypercubes Q_1, Q_2, Q_3 that were sketched above in Figure 7.1. We can label their vertices by binary numbers as in Figure 7.2.

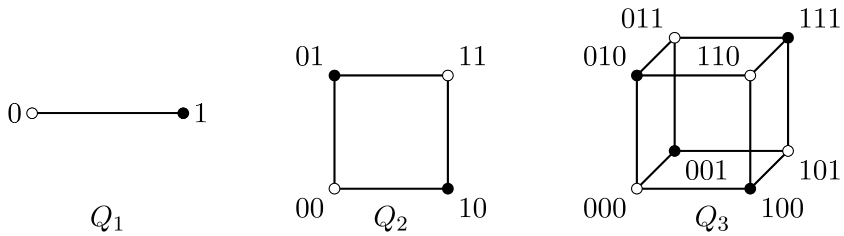


Figure 7.2.: The hypercubes Q_1, Q_2 and Q_3 with vertices labeled by binary numbers

The following is an immediate observation.

Proposition 7.1.4. *Two distinct vertices $x_1, x_2 \in U_n \cup V_n$ of the hypercube Q_n have either zero or two common neighbors $y_1, y_2 \in U_n \cup V_n$. The latter is true if and only if x_1 and x_2 differ in exactly two binary digits. In this case the subgraph $Q_n(x_1, x_2, y_1, y_2) \subset Q_n$ induced by the vertices x_1, x_2, y_1, y_2 is isomorphic to the square Q_2.*

In particular, we have $K_{2,3} \not\subset Q_n$.

Notation 7.1.5. Let $n \in \mathbb{N}$ and $k < n$.

1. We define $Q_n^{(+k)} \subset Q_n$ as the subgraph induced by the vertices $x \in U_n \cup V_n$ with $[x]_k = 1$, and we let $Q_n^{(-k)}$ be the subgraph induced by the vertices $x \in U_n \cup V_n$ with $[x]_k = 0$.

2. We let $E_n^{(k)} \subset E_n$ be the set of edges $ij \in E_n$ with $i = j\#k$.

Lemma 7.1.6. *The following properties hold for the hypercubes Q_n.*

1. *For every proper sub-hypercube $Q' \subsetneq Q_n$ there is a $k < n$ such that Q' is contained in $Q_n^{(+k)}$ or $Q_n^{(-k)}$.*

2. *Consider the equivalence relation on* $E_n^{(k)}$ *generated by*

$$i_1 j_1 \sim_0 i_2 j_2 \quad :\Leftrightarrow \quad Q_n(i_1, i_2, j_1, j_2) \cong Q_2,$$

where $Q_n(i_1, i_2, j_1, j_2) \subset Q_n$ *is the subgraph induced by* i_1, i_2, j_1, j_2. *Then all edges* $ij \in E_n^{(k)}$ *are in the same equivalence class.*

3. *Assume that a subgraph* $G \subset Q_n$ *is closed under squares in the sense that* G *contains all edges of a square in* Q_n *as soon as it contains two adjacent edges of the square. Then* G *is a disjoint union of sub-hypercubes.*

Proof. Ad (1).

Assume that $Q' \cong Q_m$ for some $m < n$ and let $\varphi : \{0, \ldots, 2^m - 1\} \to \{0, \ldots, 2^n - 1\}$ be the associated embedding of Q_m into Q_n. Without loss of generality $\varphi(0) = 0$. Then, for every $\ell < m$ there is some $k_\ell < n$ such that

$$\varphi(2^\ell) = 2^{k_\ell}$$

holds. Indeed, 2^ℓ is obtained from 0 by flipping the ℓ-th binary digit, and therefore $\varphi(2^\ell)$ must be obtained from $\varphi(0) = 0$ by flipping exactly one binary digit as well, for φ respects the edge relation of the involved hypercubes. Clearly, the k_ℓ are pairwise distinct for different $\ell < m$. We claim that $\varphi(\sum_{\ell<m} \alpha_\ell 2^\ell) = \sum_{\ell<m} \alpha_\ell 2^{k_\ell}$ holds for all $\alpha_0, \ldots, \alpha_{m-1} \in \{0, 1\}$. By definition this is true for $\alpha_0 = \cdots = \alpha_{m-1} = 0$ or if exactly one of those parameters is equal to one. For an inductive argument, assume the claim is true if less than r parameters are nonvanishing, and let $(\alpha_0, \ldots, \alpha_{m-1}) \in \{0, 1\}^m$ have exactly r non-vanishing parameters with $r \geq 2$. Without loss of generality $\alpha_0 = \alpha_1 = 1$. We want to prove the claim for the vertex $v = \sum_{\ell<m} \alpha_\ell 2^\ell$ of Q_m. Consider the vertices

$$u_0 = \sum_{\ell=1,2,\ldots,m-n} \alpha_\ell 2^\ell \quad \text{and} \quad u_1 = \sum_{\ell=0,2,3,\ldots,m-n} \alpha_\ell 2^\ell$$

that are obtained from v by flipping the first and second binary digit, respectively. The original vertex v is the unique common neighbor of u_0 and u_1 in Q_m. Hence, $\varphi(v)$ is also the unique common neighbor of $\varphi(u_0)$ and $\varphi(u_1)$. For the vertices u_0 and u_1 we can assume the claim by the induction hypothesis. Then the claim for $\varphi(v)$ follows using that $\varphi(v)$ is the unique common neighbor of $\varphi(u_0)$ and $\varphi(u_1)$. Altogether, one obtains $Q' \subset Q_n^{(-k)}$ for any $k \notin \{k_\ell \mid \ell < m\}$.

Ad (2). Phrasing the equivalence relation in other terms, one has $i_1 j_1 \sim_0 i_2 j_2$ if and only if there is some $k \neq \ell < n$ such that $j_2 = i_1 \# \ell$ and $i_2 = j_1 \# \ell$. Clearly, starting from some edge $ij \in E_n^{(k)}$ one can obtain any other edge $i'j' \in E_n^{(k)}$ by successively flipping binary digits of i and j other than the k-th digit (at the same time for both i and j). Each such step corresponds to an application of $\sim_0$, and thus the statement follows.

Ad (3). Without loss of generality assume that $G \subset Q_n$ is connected. Evidently, G is a sub-hypercube of Q_n if its vertex set $\{i < 2^n \mid i \in G\}$ is closed under flipping k-th binary digits where k ranges over a set $K \subset \{0, \ldots, n-1\}$ of indices. Let

$$K := \{k < n \mid \exists x \in G : x \# k \in G\},$$

and let $i \in G$ be arbitrary. We need to show that $i\#k$ is in G as well. By the definition of K there exists some $j \in G$ with $j\#k \in G$, and since G is connected there is a path $i_1 \dots i_m$ in G which connects $i = i_1$ and $j = i_m$.

The assumption from the statement tells us that $(x\#k)\#\ell$ is in G whenever $x \in G$ and both $x\#k$ and $x\#\ell$ are in G. We prove $i\#k \in G$ by induction over m. If $m = 1$ there is nothing to show. Otherwise, one has without loss of generality $i_m = i_{m-1}\#\ell$ for some $\ell \neq k$. It follows

$$i_{m-1}\#k = (i_m\#\ell)\#k \in G,$$

i.e. the vertex i_{m-1} already has the property we required from i_m. Hence, the statement follows by induction. $\square$

7.2. Hypercube C*-algebras are subhomogeneous

In this section we prove that every hypercube C*-algebra $C^*(Q_n)$ is subhomogeneous, and more particularly, for every irreducible representation ρ of $C^*(Q_n)$ and projection p_x we have that $\rho(p_x)$ is either zero or a rank-one projection. We prove this by analyzing paths in the hypercube with the aim of showing

$$p_\mu p_\nu = p_\nu p_\mu$$

for any two loops μ and ν in Q_n.

To do that, let us describe a path μ in Q_n in another way. Given $\mu = x_1 \dots x_{m+1}$ with vertices $x_1, \dots, x_{m+1} \in U_n \cup V_n$ one can choose indices $k_1, \dots, k_m < n$ such that

$$x_{i+1} = x_i\#k_i$$

holds for all $i \leq m$. In operational terms, the index k_i records in which direction the path μ continues at the vertex x_i. It is possible to recover the whole path μ from the starting vertex x_1 and the sequence $k_1 \dots k_m$. Therefore, we may use the following notation.

Notation 7.2.1. Given a vertex $x \in U_n \cup V_n$ and a sequence of indices $k_1, \dots, k_m < n$ we write

$$[x;\, k_1 \dots k_m]$$

for the path $\mu = x_1 \dots x_{m+1}$ given by

$$x_1 = x, \qquad \text{and} \qquad x_{i+1} = x_i\#k_i \quad \text{for } i \leq m.$$

Proposition 7.2.2. *1. For every path μ in Q_n there is a unique vertex $x \in U_n \cup V_n$ and a unique sequence $k_1 \dots k_m$ of indices with $k_i < n$ such that*

$$\mu = [x;\, k_1 \dots k_m].$$

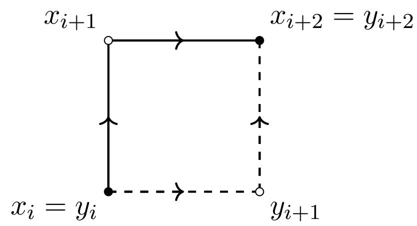


Figure 7.3.: The subgraph of Q_n induced by $x_{i-1}, x_i, x_{i+1}, x_i'$

2. *The path $[x;\, k_1 \dots k_m]$ is a loop if and only if for every $\ell < n$ the set $\{i \le m \mid k_i = \ell\}$ has an even number of elements.*

Proof. The first statement follows from the definition of edges in Q_n. The second statement is true since the ℓ-th binary digit of the endpoint of $[x;\, k_1 \dots k_m]$ is equal to $[x]_\ell + |\{i \le m \mid k_i = \ell\}| \mod 2$. $\square$

Lemma 7.2.3. *Let $\mu = [x;\, k_1 \dots k_i k_{i+1} \dots k_m]$ be a path in Q_n as above, and obtain*

$$\mu = [x;\, k_1 \dots k_{i-1} k_{i+1} k_i k_{i+2} \dots k_m]$$

by swapping the elements k_i and k_{i+1} in the sequence $k_1 \dots k_m$ for some $i < m$.

1. *If $k_i \neq k_{i+1}$, then $p_\mu = -p_\nu$.*
2. *If $k_i = k_{i+1}$, then $p_\mu = p_\nu$.*

Proof. If $k_i = k_{i+1}$, then $\mu = \nu$ and the statement is trivial. So, let us assume $k_i \neq k_{i+1}$. Write $\mu = x_1 \dots x_{m+1}$ and $\nu = y_1 \dots y_{m'+1}$ for vertices $x_j, y_j \in U_n \cup V_n$. Clearly,

$$x_j = y_j \quad \text{for all } j \le i.$$

Further, observe that

$$x_{i+2} = (x_i \# k_i) \# k_{i+1} = (x_i \# k_{i+1}) \# k_i = (y_i \# k_{i+1}) \# k_i = y_{i+2},$$

since it makes no difference in which order binary digits are flipped. Hence, we have as well

$$x_j = y_j \quad \text{for all } j \ge i+2.$$

Let us compare the elements $p_{x_i} p_{x_{i+1}} p_{x_{i+2}}$ and $p_{y_i} p_{y_{i+1}} p_{y_{i+2}}$. First, note that the terms differ only in the middle factor, for $x_i = y_i$ and $x_{i+2} = y_{i+2}$. One readily checks that the involved vertices form a subgraph of Q_n isomorphic to the square as in Figure 7.3.

In view of Proposition 7.1.4 the vertices x_{i+1} and y_{i+1} are the only common neighbors of x_i and x_{i+1}. Therefore, by (GP2) we have

$$p_{x_i} p_z p_{x_{i+2}} = 0$$

for every vertex $z \in (U_n \cup V_n) \setminus \{x_{i+1}, y_{i+1}\}$. With (GP1) it follows

$$\begin{aligned} p_{x_i} p_{x_{i+1}} p_{x_{i+2}} &= p_{x_i} \left(\sum_{z \in U_n \cup V_n} p_z - p_{y_{i+1}} \right) p_{x_{i+2}} \\ &= p_{x_i} p_{x_{i+2}} - p_{x_i} p_{y_{i+1}} p_{x_{i+2}}. \end{aligned}$$

The first term in the sum on the right-hand side vanishes by (GP1). Hence, we obtain

$$p_{x_i} p_{x_{i+1}} p_{x_{i+2}} = -p_{x_i} p_{y_{i+1}} p_{x_{i+2}} = -p_{y_i} p_{y_{i+1}} p_{y_{i+2}},$$

and the statement follows immediately. □

Lemma 7.2.4. *Let $\mu = [x;\, k_1 \dots k_m]$ and $\nu = [x;\, \ell_1 \dots \ell_{m'}]$ be two loops in Q_n starting at $x \in U_n \cup V_n$. Then $p_\mu p_\nu = p_\nu p_\mu$.*

Proof. Evidently, we have

$$p_\mu p_\nu = p_{[x;\, k_1 \dots k_m \ell_1 \dots \ell_{m'}]}$$

as well as

$$p_\nu p_\mu = p_{[x;\, \ell_1 \dots \ell_{m'} k_1 \dots k_m]}.$$

From the previous lemma we obtain immediately

$$p_\mu p_\nu = \pm p_\nu p_\mu.$$

It only remains to check that the sign on the right-hand side of this identity is positive. To see this, note that the path $[x;\, k_1 \dots k_m]$ is a loop. Therefore, every $\ell < n$ features an even number of times in the sequence $k_1 \dots k_m$. Now, to move the index ℓ_1 from the right to the left in the combined sequence $k_1 \dots k_m \ell_1 \dots \ell_{m'}$, one needs to swap it first with k_m, then with k_{m-1}, and so on. Each time ℓ_1 is swapped with an index $k_i \neq \ell_1$ the sign of the corresponding algebra element $p_{[x;\,\cdot]}$ changes. However, since there is an even number of indices $k_i \neq \ell_1$ this sign change occurs an even number of times. Hence, in the end we have

$$p_{[x;\, k_1 \dots k_m \ell_1 \dots \ell_{m'}]} = p_{[x;\, \ell_1 k_1 \dots k_m \ell_2 \dots \ell_{m'}]}.$$

Repeating this argument successively for the indices $\ell_2, \dots, \ell_{m'}$ yields the claim. □

Theorem 7.2.5. *The C^*-algebra $C^*(Q_n)$ is 2^{n-1}-subhomogeneous, and one has for all vertices $x \in U_n \cup V_n$*

$$\dim(\rho(p_x)\mathcal{H}) \leq 1 \tag{7.1}$$

for every irreducible representation ρ of $C^*(Q_n)$ on a Hilbert space $\mathcal{H}$. We call such a representation rank-one.

Proof. Let $x \in U_n \cup V_n$. By Proposition 4.2.3 the corner $p_x C^*(Q_n) p_x$ is the closed span of all elements p_μ, where μ is a loop in Q_n that starts at x. By Lemma 7.2.4 the elements p_μ commute pairwise. Consequently, the corner $p_x C^*(Q_n) p_x$ is a commutative C*-algebra. From this, Equation (7.1) follows immediately with general C*-algebra arguments, see e.g. [10, p. IV.1.1.7]. Evidently, then $C^*(Q_n)$ is 2^{n-1}-subhomogeneous, for $1 = \sum_{x \in U_n} p_x$ and $|U_n| = 2^{n-1}$. □

Corollary 7.2.6. *The C*-algebra $C^*(Q_n)$ is nuclear for all $n \in \mathbb{N} \setminus \{0\}$.*

Remark 7.2.7. In view of Problem 7.0.2 we would like to show more generally for any graph G with $K_{2,3} \not\subset G$ that the corners $p_x C^*(G) p_x$ are commutative. However, the arguments used in the proof of the previous theorem and its preceding lemmas rely heavily on the particular structure of Q_n as a hypercube, and they cannot be easily generalized.

7.3. Rank-one representations and edge-weightings

In this section we describe the rank-one representations of $C^*(Q_n)$ using certain complex edge weightings of Q_n. This will provide the necessary tools for classifying all irreducible representations of $C^*(Q_n)$ in the next section.

Recall that we call a representation ρ of $C^*(Q_n)$ rank-one if every projection $\rho(p_x)$ for $x \in U_n \cup V_n$ is either zero or a rank-one projection. In view of Theorem 7.2.5, every irreducible representation of $C^*(Q_n)$ is rank-one.

Definition 7.3.1. Let $c : E_n \to \mathbb{C}$ be a complex weighting of the edges of Q_n and let

$$Q_n(c) = (U_n(c), V_n(c), E_n(c))$$

be the subgraph of Q_n induced by the edges $ij \in E_n$ with $c(ij) \neq 0$. We call the weighting c *admissible* if it satisfies

$$\sum_{i \in \mathcal{N}(j_1) \cap \mathcal{N}(j_2)} c(ij_1)\overline{c(ij_2)} = \delta_{j_1 j_2} \qquad \text{for all } j_1, j_2 \in V_n(c), \tag{7.2}$$

as well as

$$\sum_{j \in \mathcal{N}(i_1) \cap \mathcal{N}(i_2)} c(i_1 j)\overline{c(i_2 j)} = \delta_{i_1 i_2} \qquad \text{for all } i_1, i_2 \in U_n(c), \tag{7.3}$$

where $\mathcal{N}(x)$ with $x \in U_n \cup V_n$ denotes the set of neighbors of the vertex x in Q_n.

The following lemma is needed to construct rank-one representations from admissible edge weightings. It will also be crucial later when we exploit the description of rank-one representations by edge weightings to classify all irreducible representations of $C^*(Q_n)$.

Lemma 7.3.2. *Let $c : E_n \to \mathbb{C}$ be an admissible edge weighting of the hypercube Q_n, i.e. c satisfies* (7.2) *and* (7.3). *Further, assume that $x_1, x_2, y_1, y_2 \in U_n \cup V_n$ are four vertices which induce a subgraph of Q_n that is isomorphic to a square as in Figure 7.3.*

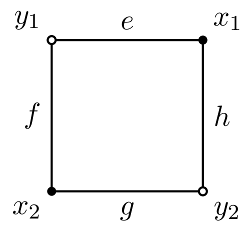


Figure 7.4.: A square in the hypercube Q_n with edges e, f, g, h

1. *If $c(e) \neq 0$ and $c(f) \neq 0$, then we have $c(g) \neq 0$ and $c(h) \neq 0$ as well, and*
$$\frac{c(e)}{c(f)} = -\overline{\left(\frac{c(g)}{c(h)}\right)}.$$
2. *In that case, we also have*
$$|c(e)| = |c(g)| \quad \textit{and} \quad |c(f)| = |c(h)|.$$
3. *The induced subgraph $Q_n(c) \subset Q_n$ is a disjoint union of sub-hypercubes.*

Proof. Ad (1). The vertices y_1 and y_2 have exactly two common neighbors in Q_n, namely x_1 and x_2, see Proposition 7.1.4. Therefore, condition (7.2) or (7.3) for weightings gives
$$c(e)\overline{c(h)} + c(f)\overline{c(g)} = 0.$$
The statement follows immediately by a standard calculation.

Ad (2). The previous statement entails
$$\left|\frac{c(e)}{c(f)}\right| = \left|\frac{c(g)}{c(h)}\right|,$$
where all terms $c(e), c(f), c(g), c(h)$ are non-zero by assumption and (1). Similarly, one obtains
$$\left|\frac{c(f)}{c(g)}\right| = \left|\frac{c(h)}{c(e)}\right|.$$
It follows
$$|c(e)| = |c(g)| \left|\frac{c(f)}{c(g)}\right| \cdot \left|\frac{c(e)}{c(f)}\right| = |c(g)| \left|\frac{c(h)}{c(e)}\right| \cdot \left|\frac{c(g)}{c(h)}\right| = \frac{|c(g)|^2}{|c(e)|}.$$
Therefore, $|c(e)|^2 = |c(g)|^2$, and the claim follows.

Ad (3). In view of (1), $Q_n(c)$ has the following property: Whenever it contains two adjacent edges of a square in Q_n, then it contains all four edges of the square. Thus, the claim follows directly from Lemma 7.1.6(3).

□

Definition 7.3.3. Given an admissible edge weighting $c : E_n \to \mathbb{C}$ of Q_n, let

$$\rho_c : C^*(Q_n) \to M_{U_n(c)} \cong B(\mathbb{C}^{U_n(c)})$$

be the unique representation that extends the assignment

$$\begin{aligned} &p_i \mapsto E_{ii} \in M_{U_n(c)}, && i \in U_n(c), \\ &p_j \mapsto \Big[c(i_1 j)\overline{c(i_2 j)}\Big]_{i_1, i_2 \in U_n} \in M_{U_n(c)}, && j \in V_n(c), \\ &p_x \mapsto 0, && x \in (U_n \cup V_n) \setminus (U_n(c) \cup V_n(c)). \end{aligned}$$

We say that the representation ρ_c is induced by the weighting c.

Proposition 7.3.4. *The representation ρ_c in the previous definition is well-defined. In particular, for every $j \in V_n(c)$ the matrix $\rho_c(p_j)$ is the projection onto the span of the vector*

$$\psi_j = [c(ij)]_{i \in U_n(c)} \in \mathbb{C}^{U_n(c)}.$$

Proof. The second claim is true since the vector ψ_j is normalized due to (7.2). Thus, the elements $\rho_c(p_x)$ are projections for all $x \in U_n \cup V_n$. We need to check that the operators $\rho_c(p_x)$ for $x \in U_n(c) \cup V_n(c)$ satisfy the relations (GP1) and (GP2) from Definition 4.2.1. First, let us note that $|U_n(c)| = |V_n(c)|$ holds since by Lemma 7.3.2(3) the graph $Q_n(c)$ is a disjoint union of sub-hypercubes.

Evidently, the E_{ii} form a partition of unity. At the same time (7.2) guarantees that the vectors ψ_j are orthogonal to each other. What's more, because of $|V_n(c)| = |U_n(c)|$ the vectors ψ_j form an orthogonal basis of $\mathbb{C}^{U_n(c)}$. Hence, the projections onto the spans of the ψ_j form a partition of unity in $B(\mathbb{C}^{U_n(c)})$ as well. Further, one readily checks that ψ_j is orthogonal to $e_i \in \mathbb{C}^{U_n(c)}$ if $ij \notin E_n$ since the i-th entry of ψ_j vanishes. Thus, the universal property of $C^*(Q_n)$ yields the desired representation ρ_c. □

Let us shortly summarize the notation that was introduced so far. Given an (admissible) edge weighting $c : E_n \to \mathbb{C}$ of Q_n, we denote

- by $Q_n(c) = (U_n(c), V_n(c), E_n(c))$ the subgraph of Q_n induced by the edges where c does not vanish,
- by $\rho_c : C^*(Q_n) \to B(\mathbb{C}^{U_n(c)})$ the representation of $C^*(Q_n)$ induced by c.

Example 7.3.5. As an example, consider the square Q_2 with edges e, f, g, h as in Figure 7.3. For any $t \in [0, 1]$ and $t' := 1 - t$ an admissible edge weighting is given by

$$c(e) = \sqrt{t}, \quad c(f) = \sqrt{t'}, \quad c(g) = \sqrt{t}, \quad c(h) = -\sqrt{t'}.$$

We sketch this weighting in Figure 7.5. If $t \in (0, 1)$, then $Q_2(c) = Q_2$, but if $t = 0$ ($t = 1$), then $Q_2(c)$ is obtained from Q_2 by removing the edges e, g (f, h, resp.). In any event, $Q_2(c) \subset Q_2$ is a disjoint union of sub-hypercubes.

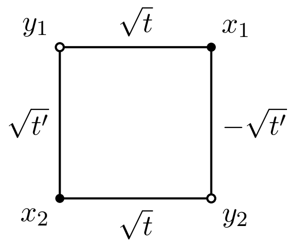


Figure 7.5.: An admissible edge weighting of Q_2

To compute the representation ρ_c let us assume that $\{x_1, x_2\} = U_2$ and $\{y_1, y_2\} = V_2$. Then, for any $t \in [0,1]$ the representation $\rho_c : C^*(Q_2) \to B(\mathbb{C}^2)$ is given

$$\rho_c(p_{x_1}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \rho_c(p_{y_1}) = \begin{pmatrix} t & \sqrt{tt'} \\ \sqrt{tt'} & t' \end{pmatrix} = \begin{pmatrix} \sqrt{t} & \sqrt{t'} \end{pmatrix} \begin{pmatrix} \sqrt{t} \\ \sqrt{t'} \end{pmatrix},$$
$$\rho_c(p_{x_2}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \rho_c(p_{y_2}) = \begin{pmatrix} t' & -\sqrt{tt'} \\ -\sqrt{tt'} & t \end{pmatrix} = \begin{pmatrix} -\sqrt{t'} \\ \sqrt{t} \end{pmatrix} \begin{pmatrix} -\sqrt{t'} & \sqrt{t} \end{pmatrix}.$$

Lemma 7.3.6. *Let $c : E_n \to \mathbb{C}$ be an admissible weighting of the hypercube Q_n, and let $Q_n(c) \subset Q_n$ be as in Definition 7.3.1. Then the following holds:*

1. *Assume that $Q_n(c) = \bigcup_\ell Q_n^{(\ell)}$ for disjoint sub-hypercubes $Q_n^{(\ell)} \subset Q_n$. Then the representation $\rho_c : C^*(Q_n) \to B(\mathbb{C}^{U_n(c)})$ induced by c is a direct sum of irreducible representations $\rho^{(\ell)}$ which are vanishing on p_x for $x \notin Q_n^{(\ell)}$, respectively. In particular, ρ_c is irreducible if and only if $Q_n(c)$ is a single sub-hypercube of Q_n.*

2. *If $c' : E_n \to \mathbb{C}$ is another admissible weighting of Q_n, then the representations ρ_c and $\rho_{c'}$ are unitarily equivalent if and only if $Q_n(c') = Q_n(c)$ and there are numbers $\lambda_x \in S^1$ for $x \in U_n(c) \cup V_n(c)$ such that*
$$c'(ij) = \lambda_i \lambda_j c(ij) \qquad \textit{for all } ij \in E_n(c).$$

3. *For any rank-one representation $\rho : C^*(Q_n) \to B(\mathcal{H})$ of the hypercube algebra $C^*(Q_n)$ on a Hilbert space $\mathcal{H}$ there exists an admissible weighting $c : E_n \to \mathbb{C}$ such that ρ is unitarily equivalent to the representation ρ_c.*

Proof. Ad (1). First, let $U_n^{(\ell)} := U_n \cap Q_n^{(\ell)} \subset U_n(c)$ be the set of vertices from $U_n(c)$ contained in $Q_n^{(\ell)}$ for all ℓ. Then the partition

$$U_n(c) = \bigcup_\ell U_n^{(\ell)}$$

yields a natural block matrix form for the matrices in $M_{U_n(c)} = B(\mathbb{C}^{U_n(c)})$. In particular, the diagonal blocks correspond to the sub-hypercubes $Q_n^{(\ell)}$. We claim that $\rho_c(p_j)$ vanishes outside the diagonal block corresponding to the sub-hypercube that contains j.

Indeed, one readily checks that $\rho_c(p_j)$ has a non-zero entry in the (i,i')-th position if and only if $ij \in E_n(c)$ and $i'j \in E_n(c)$. This is the case if and only if both i and i' are contained in the same sub-hypercube $Q_n^{(\ell)}$ as the vertex j. Consequently, $\rho_c(p_j)$ is non-vanishing only on the diagonal block corresponding to the sub-hypercube $Q_n^{(\ell)}$ that contains j. It follows that ρ_c is a direct sum of representations $\rho^{(\ell)}$ which are vanishing on p_x for $x \notin Q_n^{(\ell)}$, respectively.

Further, if $Q_n(c)$ consists of a single sub-hypercube, then for any two vertices $i, i' \in U_n(c)$ there is a path $x_1x_2\dots x_m$ in $Q_n(c)$ which connects i and i'. One checks that the matrix

$$\rho(p_{x_1})\rho(p_{x_2})\cdots\rho(p_{x_m})$$

has a single non-zero entry in the (i,i')-th position. Hence, $\rho_c(C^*(Q_n)) = M_{U_n(c)}$ and the representation is irreducible.

Ad (2). If the representations ρ_c and $\rho_{c'}$ are unitarily equivalent, then the sets of vertices $x \in U_n \cup V_n$ with $\rho_c(p_x) \neq 0$ and $\rho_{c'}(p_x) \neq 0$, respectively, coincide. Consequently, $U_n(c) = U_n(c')$. Let $\mathcal{U} \in B(\mathbb{C}^{U_n(c)})$ be a unitary matrix such that

$$\rho_{c'}(\cdot) = \mathcal{U}\rho_c(\cdot)\mathcal{U}^*$$

holds. Because of $\rho_{c'}(p_i) = E_{ii} = \rho_c(p_i)$ for all $i \in U_n(c)$, the unitary $\mathcal{U}$ must be diagonal, and there are numbers $\lambda_i \in S^1$ such that

$$\mathcal{U} = \operatorname{diag}((\lambda_i)_{i\in U_n(c)}).$$

It is not hard to see that $\mathcal{U}\rho_c(p_j)\mathcal{U}^*$ is the projection onto the span of $\mathcal{U}\psi_j$ for all $j \in V_n(c)$ where $\psi_j = (c(ij))_{i\in U_n(c)}$ is the vector from Proposition 7.3.4. Let $\psi'_j = (c'(ij))_{i\in U_n(c)} \in \mathbb{C}^{U_n(c)}$ for $j \in V_n(c)$ be the vectors such that $\rho_{c'}(p_j)$ is the projection onto the span of ψ'_j. Then, $\mathcal{U}\rho_c(p_j)\mathcal{U}^* = \rho_{c'}(p_j)$ implies that the vectors

$$\mathcal{U}\psi_j = (\lambda_i c(ij))_{i\in U_n(c)} \quad \text{and} \quad \psi'_j = (c'(ij))_{i\in U_n(c)}$$

have the same span. This is true if and only if there are numbers $\lambda_j \in \mathbb{C}\setminus\{0\}$ for $j \in V_n(c)$ with

$$\lambda_i\lambda_j c(ij) = c'(ij) \qquad \text{for all } ij \in E_n(c).$$

Because of conditions (7.2) and (7.3) the numbers λ_j must be of modulus one as well.

This proves that unitarily equivalent induced representations ρ_c and $\rho_{c'}$ must satisfy the condition from the statement. Conversely, it is easy to see that this condition implies unitary equivalence of ρ_c and $\rho_{c'}$.

Ad (3). Assume that $\rho : C^*(Q_n) \to B(\mathcal{H})$ is a rank-one representation of the hypercube algebra $C^*(Q_n)$ on a Hilbert space $\mathcal{H}$. Let $Q_n(\rho) = (U_n(\rho), V_n(\rho), E_n(\rho)) \subset Q_n$ be the subgraph induced by the vertices $x \in U_n \cup V_n$ with $\rho(p_x) \neq 0$. By applying a unitary transformation if necessary we may assume $\mathcal{H} = \mathbb{C}^{U_n(\rho)}$ and $\rho(p_i) = E_{ii}$ for all

$i \in U_n(\rho)$. For $j \in V_n(\rho)$, $\rho(p_j)$ is the projection onto the span of some normalized vector $\psi_j \in \mathbb{C}^{U_n(\rho)}$. Let a weighting $c : E_n \to \mathbb{C}$ be given by

$$c(ij) := \begin{cases} [\psi_j]_i, & \text{if } \rho(p_j) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $[\psi_j]_i = 0$ if $ij \notin E_n$ since in this case $\rho(p_j)$ and $\rho(p_i)$ are orthogonal. One has $|U_n(\rho)| = |V_n(\rho)|$ for both the $\rho(p_i)$ and $\rho(p_j)$ for $i \in U_n, j \in V_n$ form a partition of unity in $M_{U_n(\rho)}$. Hence, the ψ_j form an orthonormal basis of $\mathbb{C}^{U_n(\rho)}$ and satisfy (7.2). The matrix which contains the ψ_j as columns is, thus, unitary, and it follows that its rows form an orthonormal basis of $\mathbb{C}^{U_n(\rho)}$ as well. Therefore, (7.3) is satisfied, and c is an admissible weighting of Q_n. □

7.4. Rank-one representations induced by points in a simplex

So far we have seen that every irreducible representation of $C^*(Q_n)$ is rank-one, and that every rank-one representation is induced by an admissible edge weighting of Q_n. In the end, we want to classify all irreducible representations of $C^*(Q_n)$ up to unitary equivalence. Admissible edge weightings are not the right solution for this problem, for two different edge weightings can define unitarily equivalent representations. Therefore, we introduce below a particular class of edge weightings $c_{\mathbf{t}}$ depending on points $\mathbf{t}$ in the standard simplex

$$\Delta_{n-1} = \left\{ [t_0, \dots, t_{n-1}] \mid t_0, \dots, t_{n-1} \geq 0, \sum_i t_i = 1 \right\}.$$

The definition below is a straight generalization of the edge weightings of Q_2 from Example 7.3.5.

Definition 7.4.1. For every tuple $\mathbf{t} = [t_0, \dots, t_{n-1}] \in \Delta_{n-1}$ let an edge weighting $c_{\mathbf{t}}$ of Q_n be given by

$$c_{\mathbf{t}}(ij) = (-1)^{\mathrm{par}_k(i)} \sqrt{t_k} \qquad \text{for all } ij \in E_n,$$

where $k < n$ is the unique number with $j = i\#k$ and $\mathrm{par}_k(i)$ is the sum of the first $(k+1)$-binary digits of i modulo 2 as in Notation 7.1.1. Let

$$\rho_{\mathbf{t}} : C^*(Q_n) \to M_{U_n}$$

be the representation of $C^*(Q_n)$ induced by the weighting $c_{\mathbf{t}}$.

Proposition 7.4.2. *The representation $\rho_{\mathbf{t}}$ in the previous definition is well-defined, i.e. the edge weighting $c_{\mathbf{t}}$ is admissible for all $\mathbf{t} \in \Delta_{n-1}$ and its induced subgraph $Q_n(c_{\mathbf{t}})$ satisfies $U_n(c_{\mathbf{t}}) = U_n$.*

Proof. Let us first check $U_n(c_{\mathbf{t}}) = U_n$. Since $\mathbf{t} = [t_0, \dots, t_{n-1}] \in \Delta_{n-1}$ satisfies $\sum_k t_k = 1$, there exists at least one index ℓ with $t_\ell \neq 0$. Consequently, for every $i \in U_n$ one has $c_{\mathbf{t}}(ij) \neq 0$ if one chooses $j := i\#\ell$. Thus, i is contained in the subgraph $Q_n(c_{\mathbf{t}})$, and it follows that $U_n(c_{\mathbf{t}}) = U_n$.

It remains to check that $c_{\mathbf{t}}$ satisfies the conditions (7.2) and (7.3) from Definition 7.3.1. One immediately observes for any fixed $j \in V_n$ that

$$\sum_{i \in U_n} |c_{\mathbf{t}}(ij)|^2 = \sum_{k=0}^{n-1} t_k = 1,$$

and similarly for any fixed $i \in U_n$ that

$$\sum_{j \in V_n} |c_{\mathbf{t}}(ij)|^2 = \sum_{k=0}^{n-1} t_k = 1.$$

On the other hand, for distinct $j_1, j_2 \in U_n$ recall from Proposition 7.1.4 that there are either no or exactly two common neighbors $i_1, i_2 \in \mathcal{N}(j_1) \cap \mathcal{N}(j_2)$. In the latter case let $\ell < k$ be such that $j_1 = i_1\#\ell$, $j_2 = i_2\#\ell$ as well as $j_1 = i_2\#k$ and $j_2 = i_1\#k$ hold. Then we have

$$\begin{aligned} &c(i_1 j_1)\overline{c(i_1 j_2)} + c(i_2 j_1)\overline{c(i_2 j_2)} \\ &\qquad = (-1)^{\mathrm{par}_\ell(i_1) + \mathrm{par}_k(i_1)} \sqrt{t_\ell}\sqrt{t_k} + (-1)^{\mathrm{par}_k(i_2) + \mathrm{par}_\ell(i_2)} \sqrt{t_k}\sqrt{t_\ell}. \end{aligned}$$

One checks that the two terms in this sum have opposite signs, and therefore the sum vanishes. Indeed, the binary representation of i_1 and i_2 differs exactly in the ℓ-th and k-th digit. As $\ell < k$ it follows that $\mathrm{par}_\ell(i_1) \neq \mathrm{par}_\ell(i_2)$ and $\mathrm{par}_k(i_1) = \mathrm{par}_k(i_2)$. Thus, (7.2) is satisfied. The other condition (7.3) can be checked in the same way. □

Example 7.4.3. For $n = 3$ the vertices in U_3 are labeled by the binary numbers $000, 011, 101$ and 110. Similarly, the vertices in V_3 are labeled by $001, 010, 100, 111$. Thus, for $\mathbf{t} = [a^2, b^2, c^2] \in \Delta_2$ one verifies that $\rho_{\mathbf{t}}(p_j)$ for $j \in V_3$ is given by the projection onto $\psi_j \in \mathbb{C}^{U_3}$, respectively, where

$$\psi_{001} = \begin{pmatrix} a \\ b \\ c \\ 0 \end{pmatrix}, \quad \psi_{010} = \begin{pmatrix} b \\ -a \\ 0 \\ c \end{pmatrix}, \quad \psi_{100} = \begin{pmatrix} c \\ 0 \\ -a \\ -b \end{pmatrix}, \quad \psi_{111} = \begin{pmatrix} 0 \\ c \\ -b \\ a \end{pmatrix}.$$

Here, we identify $\mathbb{C}^{U_3}$ with $\mathbb{C}^4$ via

$$e_{000} := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_{011} := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_{101} := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_{110} := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

At the end of this section we prove a technical lemma which describes $\rho_{\mathbf{t}}$ for a boundary point $\mathbf{t} \in \Delta_{n-1}$ as a direct sum of smaller representations $\rho_{\mathbf{s}}$ for suitable $\mathbf{s} \in \Delta_{n-2}$. This will be useful in the proof of Theorem 7.5.3 where the irreducible representations of $C^*(Q_n)$ are classified. To state the lemma let us introduce some notation.

Notation 7.4.4. For $\ell < n$ and $y = \sum_{k=0}^{n-2} y_k 2^k < 2^{n-1}$ with $y_k \in \{0,1\}$, let

$$y^{(+\ell)} := \sum_{k=0}^{\ell-1} y_k 2^k + 2^\ell + \sum_{k=\ell}^{n-2} y_k 2^{k+1} < 2^n,$$
$$y^{(-\ell)} := \sum_{k=0}^{\ell-1} y_k 2^k + \sum_{k=\ell}^{n-2} y_k 2^{k+1} < 2^n,$$

be the numbers obtained from y by inserting a 1 or a 0, respectively, before the ℓ-th position in the binary representation of y.

Further, let $\pi^{(+\ell)}, \pi^{(-\ell)} : C^*(Q_n) \to C^*(Q_{n-1})$ be surjective $*$-homomorphisms given by

$$\pi^{(+\ell)} : p_x \mapsto \begin{cases} \tilde{p}_y, & \exists y \in U_{n-1} \cup V_{n-1} : x = (y\#0)^{(+\ell)}, \\ 0, & \text{otherwise}, \end{cases}$$
$$\pi^{(-\ell)} : p_x \mapsto \begin{cases} \tilde{p}_y, & \exists y \in U_{n-1} \cup V_{n-1} : x = y^{(-\ell)}, \\ 0, & \text{otherwise}. \end{cases}$$

Here we write $\tilde{p}_y$ for the canonical generators of $C^*(Q_{n-1})$ to distinguish them from the generators p_x of $C^*(Q_n)$.

Note that one has

$$U_n = \{y^{(-\ell)} \mid y \in U_{n-1}\} \cup \{(y\#0)^{(+\ell)} \mid y \in U_{n-1}\}$$

for all $\ell < n$. In fact, a number $x < 2^n$ is in U_n iff its binary representation contains an even number of 1s. If $x < 2^n$ is in U_n, then it has either a 0 in the ℓ-th position (in which case $x = y^{(-\ell)}$ for a suitable $y \in U_{n-1}$) or it has a 1 in the ℓ-th position (in which case $x = (y\#0)^{(+\ell)}$ for a suitable $y \in U_{n-1}$).

The $*$-homomorphisms $\pi^{(+\ell)}$ and $\pi^{(-\ell)}$ exist by the universal property of $C^*(Q_n)$.

Lemma 7.4.5. *For $\mathbf{s} = [s_0, \dots, s_{n-2}] \in \Delta_{n-2}$ and $\ell < n$ let $\mathbf{t} \in \Delta_{n-1}$ be given by*

$$\mathbf{t} = [t_0, \dots, t_{n-1}] := [s_0, \dots, s_{\ell-1}, 0, s_\ell, \dots, s_{n-2}].$$

Then one has the unitary equivalence

$$\rho_{\mathbf{t}} \cong \left(\rho_{\mathbf{s}} \circ \pi^{(+\ell)}\right) \oplus \left(\rho_{\mathbf{s}} \circ \pi^{(-\ell)}\right).$$

Proof. The representation $\rho_{\mathbf{t}}$ acts on $\mathbb{C}^{U_n}$, while the direct sum on the right-hand side of the equation in the statement acts on $\mathbb{C}^{U_{n-1}} \oplus \mathbb{C}^{U_{n-1}}$. Consider the unitary operator $\mathcal{U} : \mathbb{C}^{U_{n-1}} \oplus \mathbb{C}^{U_{n-1}} \to \mathbb{C}^{U_n}$ given by

$$\mathcal{U} : (e_x, 0) \mapsto e_{(x\#0)^{(+\ell)}}, \qquad (0, e_x) \mapsto e_{x^{(-\ell)}},$$

for all $x \in U_{n-1}$. Let us determine the representation

$$\sigma(\cdot) := \mathcal{U}\left(\left(\rho_{\mathbf{s}} \circ \pi^{(+\ell)}(\cdot)\right) \oplus \left(\rho_{\mathbf{s}} \circ \pi^{(-\ell)}(\cdot)\right)\right)\mathcal{U}^*$$

of $C^*(Q_n)$ on $\mathbb{C}^{U_n}$. For $x \in U_n$ with $[x]_\ell = 1$ we have $x = (y\#0)^{(+\ell)}$ for a suitable (and unique) $y \in U_{n-1}$. Then, $\pi^{(+\ell)}(p_x) = \tilde{p}_y$ and $\pi^{(-\ell)}(p_x) = 0$. Thus, one easily checks

$$\sigma(p_x) = \mathcal{U}\begin{pmatrix} \tilde{E}_{yy} & 0 \\ 0 & 0 \end{pmatrix}\mathcal{U}^* = E_{xx},$$

where we write a bounded operator on $\mathbb{C}^{U_{n-1}} \oplus \mathbb{C}^{U_{n-1}}$ as a 2×2 block matrix in a natural way, and $\tilde{E}_{yy} \in M_{U_{n-1}}$ is the usual standard matrix unit. One obtains

$$\sigma(p_x) = E_{xx}$$

for $x \in U_n$ with $[x]_\ell = 0$ similarly.

Next, let $x \in V_n$ with $[x]_\ell = 1$, and let $y \in V_{n-1}$ be the unique vertex with $x = (y\#0)^{(+\ell)}$. Then $\pi^{(+\ell)}(p_x) = \tilde{p}_y$ and $\pi^{(-\ell)}(p_x) = 0$. Further, $\tilde{p}_y$ is the projection onto the span of the vector $\tilde{\psi}_y \in \mathbb{C}^{U_{n-1}}$ with entries

$$[\tilde{\psi}_y]_i = \begin{cases} (-1)^{\mathrm{par}_k(i)}\sqrt{s_k}, & \exists\, k < n-1 : y = i\#k, \\ 0, & \text{otherwise}, \end{cases}$$

where i ranges over U_{n-1}. Consequently, $\sigma(p_x)$ is the projection onto the span of the vector $\varphi_x := \mathcal{U}(\tilde{\psi}_y, 0) \in \mathbb{C}^{U_n}$. Since, the second component of $(\tilde{\psi}_y, 0)$ vanishes it is clear that $[\varphi_x]_i$ vanishes for $i \in U_n$ with $[i]_\ell = 0$. For the other indices $i \in U_n$ there exists a unique $\tilde{i} \in U_{n-1}$ with $i = (\tilde{i}\#0)^{(+\ell)}$, and one computes

$$[\varphi_x]_i = \begin{cases} (-1)^{\mathrm{par}_k(\tilde{i})}\sqrt{s_k} & \exists\, k < n-1 : y = \tilde{i}\#k, \\ 0 & \text{otherwise}. \end{cases}$$

One checks that $y = \tilde{i}\#k$ is equivalent to

$$\begin{aligned} x &= (y\#0)^{(+\ell)} = (\tilde{i}\#0)^{(+\ell)}\#k = i\#k, && \text{if } k < \ell, \\ x &= (y\#0)^{(+\ell)} = (\tilde{i}\#0)^{(+\ell)}\#(k+1) = i\#(k+1), && \text{if } k \geq \ell. \end{aligned}$$

Further,

$$\mathrm{par}_k(\tilde{i}) = \begin{cases} -\mathrm{par}_k((\tilde{i}\#0)^{(+\ell)}) = -\mathrm{par}_k(i) & \text{if } k < \ell, \\ \mathrm{par}_{k+1}((\tilde{i}\#0)^{(+\ell)}) = \mathrm{par}_{k+1}(i) & \text{if } k \geq \ell, \end{cases}$$

Using $\mathbf{t} = [s_0, \dots, s_{\ell-1}, 0, s_\ell, \dots, s_{n-2}]$ we thus obtain

$$[\varphi_x]_i = \begin{cases} -(-1)^{\mathrm{par}_k(i)}\sqrt{t_k} & \exists\, k < \ell : x = i\#k, \\ (-1)^{\mathrm{par}_{k+1}(i)}\sqrt{t_{k+1}} & \exists\, k \geq \ell : x = i\#(k+1), \\ 0 & \text{otherwise.} \end{cases}$$

Analogously, one checks for $x \in V_n$ with $[x]_\ell = 0$ that $\sigma(p_x)$ is the projection onto the span of the vector $\varphi_x \in \mathbb{C}^{U_n}$ with

$$[\varphi_x]_i = \begin{cases} (-1)^{\mathrm{par}_k(i)}\sqrt{t_k} & \exists\, k < \ell : x = i\#k, \\ (-1)^{\mathrm{par}_{k+1}(i)}\sqrt{t_{k+1}} & \exists\, k \geq \ell : x = i\#(k+1), \\ 0 & \text{otherwise.} \end{cases}$$

Putting everything together and using $t_\ell = 0$, we obtain that σ is induced by the edge weighting $c : E_n \to \mathbb{C}$ with

$$c(ij) = \begin{cases} \varepsilon_j^k (-1)^{\mathrm{par}_k(i)}\sqrt{t_k} & \exists\, k < n : j = i\#k, \\ 0 & \text{otherwise ,} \end{cases}$$

for all edges $ij \in E_n$, where

$$\varepsilon_j^k = \begin{cases} -1, & \text{if } [j]_\ell = 1 \text{ and } k < \ell, \\ 1, & \text{otherwise.} \end{cases}$$

Setting $\lambda_x := (-1)^{[x]_\ell \mathrm{par}_\ell(x)}$ for all $x \in U_n \cup V_n$ it is not hard to verify

$$\lambda_i \lambda_j c(ij) = c_{\mathbf{t}}(ij),$$

where $c_{\mathbf{t}}$ is the edge weighting from Definition 7.4.1. Thus, σ is unitarily equivalent to $\rho_{\mathbf{t}}$ by Lemma 7.3.6(2). □

7.5. Irreducible representations

Building on the previous sections, we show that any irreducible representation of the hypercube algebra $C^*(Q_n)$ is unitarily equivalent to a subrepresentation of the representation $\rho_{\mathbf{t}}$ from Definition 7.4.1 for some point $\mathbf{t} = [t_0, \dots, t_{n-1}] \in \Delta_{n-1}$. We call a representation ρ_1 a subrepresentation of a representation ρ if there is a third representation ρ_2 such that $\rho = \rho_1 \oplus \rho_2$ holds.

Let us briefly outline the proof strategy. We know from the previous sections that every irreducible representation ρ is unitarily equivalent to a representation ρ_c induced by an admissible edge weighting $c : E_n \to \mathbb{C}$. Further, since ρ_c is irreducible, the induced subgraph $Q_n(c) \subset Q_n$ is a single sub-hypercube of Q_n. It will turn out that it suffices to prove $\rho_c \cong \rho_{\mathbf{t}}$ for suitable $\mathbf{t} \in \Delta_{n-1}$ in the case where $Q_n(c) = Q_n$. In this case the edge weighting c is nowhere-vanishing, and we prove the statement in two steps:

- First, in Lemma 7.5.1 we show that any nowhere-vanishing admissible weighting c satisfies $|c(ij)| = \sqrt{t_k}$ for all $ij \in E_n$ with $i = j\#k$ for a suitable point $\mathbf{t} = [t_0, \dots, t_{n-1}] \in \Delta_{n-1}$.
- Then, in Lemma 7.5.2 we prove that such a representation ρ_c is unitarily equivalent to $\rho_{\mathbf{t}}$. For that we use Lemma 7.3.6(2) which says that $\rho_c \cong \rho_{\mathbf{t}}$ holds if and only if there are $\lambda_x \in S^1$ for $x \in U_n(c) \cup V_n(c)$ such that
$$c(ij) = \lambda_i \lambda_j c_{\mathbf{t}}(ij) \qquad \text{for all } ij \in E_n(c).$$

Lemma 7.5.1. *Assume that $c : E_n \to \mathbb{C}$ is a nowhere-vanishing admissible edge weighting of the hypercube Q_n, i.e. $c(ij) \neq 0$ for all edges $ij \in E_n$.*

There is a point $\mathbf{t} = [t_0, \dots, t_{n-1}] \in \Delta_{n-1}$ such that $|c(ij)| = \sqrt{t_k}$ holds for all edges $ij \in E_n$ where $i = j\#k$.

Proof. Since c is nowhere-vanishing, we have $Q_n(c) = Q_n$. Let us set
$$t_k = |c(0[0\#k])|^2$$
for all $k < n$. Note that $\sum_{k=0}^{n-1} t_k = 1$ holds by condition (7.3) for admissible edge weightings, so that $\mathbf{t} = [t_0, \dots, t_{n-1}]$ is indeed in Δ_{n-1}.

It remains to show $|c(ij)| = \sqrt{t_k}$ for all edges $ij \in E_n$ with $i = j\#k$. Recall that we denote the set of all these edges by $E_n^{(k)}$. Now, the claim follows directly from Lemma 7.3.2(2) in combination with Lemma 7.1.6(2), since the subset of edges in $E_n^{(k)}$ with $|c(e)| = \sqrt{t_k}$ is non-empty (it contains $0[0\#k]$) and closed under taking opposite edges in a square, i.e. under the equivalence relation from Lemma 7.1.6(2). □

Lemma 7.5.2. *Let $c : E_n \to \mathbb{C}$ be a nowhere-vanishing admissible weighting of the hypercube Q_n. Then there is a tuple $\mathbf{t} = [t_0, \dots, t_{n-1}] \in \Delta_{n-1}$ such that ρ_c is unitarily equivalent to $\rho_{\mathbf{t}}$ where the latter is the representation induced by the weighting $c_{\mathbf{t}}$ from Definition 7.4.1.*

Proof. In view of Lemma 7.3.6(2) the statement is equivalent to the existence of numbers $\lambda_x \in S^1$ for $x \in U_n \cup V_n$ with
$$c(ij) = \lambda_i \lambda_j c_{\mathbf{t}}(ij) \tag{7.4}$$
for all edges $ij \in E_n$.

First, recall Lemma 7.5.1 and choose $\mathbf{t} = [t_0, \dots, t_{n-1}] \in \Delta_{n-1}$ such that
$$|c(ij)| = \sqrt{t_k} \qquad \text{for all } ij \in E_n \text{ where } i = j\#k$$
holds. We prove (7.4) for all edges $ij \in E_n^{(0)} \cup E_n^{(1)} \cup \dots \cup E_n^{(K)}$ by induction over K, where $E_n^{(k)}$ is the set of edges $ij \in E_n$ with $i = j\#k$ and $k < n$. For $K = 0$ one simply chooses
$$\lambda_i := (-1)^{\mathrm{par}_0(i)} \frac{c(ij)}{|c(ij)|} \qquad \text{for all } i \in U_n \text{ and } j := i\#0$$

as well as $\lambda_j := 1$ for all $j \in V_n$. Then (7.4) holds for all edges $ij \in E_n^{(0)}$ since

$$\lambda_i \lambda_j c_{\mathbf{t}}(ij) = (-1)^{\mathrm{par}_0(i)} \frac{c(ij)}{|c(ij)|} (-1)^{\mathrm{par}_0(i)} \sqrt{t_0} = c_{ij}$$

is true for all edges $ij \in E_n^{(0)}$.

For the induction step, we may assume without loss of generality

$$c_{\mathbf{t}}(ij) = c(ij) \qquad \text{for all } ij \in E_n^{(\ell)} \text{ for some } \ell < K.$$

Let us choose

$$\lambda_i := (-1)^{\mathrm{par}_K(i)} \frac{c(ij)}{|c(ij)|} \qquad \text{for all } i \in U_n \text{ with } [i]_K = 0 \text{ and } j := i\#K,$$

as well as

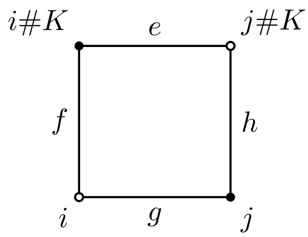
$$\lambda_j := (-1)^{\mathrm{par}_K(i)} \frac{c(ij)}{|c(ij)|} \qquad \text{for all } j \in V_n \text{ with } [j]_K = 0 \text{ and } i := j\#K.$$

For all other $i \in U_n$, $j \in V_n$ simply set $\lambda_i = \lambda_j = 1$. We first check (7.4) for all edges $ij \in E_n^{(\ell)}$ with $\ell < K$.

Indeed, if $[i]_K = 1$, then $[j]_K = 1$ holds as well and the equality is evident. Otherwise, we have both $[i]_K = 0$ and $[j]_K = 0$, where i and j differ in the ℓ-th binary digit. Therefore, one obtains

$$\lambda_i \lambda_j c_{\mathbf{t}}(ij) = (-1)^{\mathrm{par}_K(i)} \frac{c(i[i\#K])}{|c(i[i\#K])|} (-1)^{\mathrm{par}_K(j\#K)} \frac{c([j\#K]j)}{|c([j\#K]j)|} c(ij) \tag{7.5}$$

where we use the induction hypothesis to get $c_{\mathbf{t}}(ij) = c(ij)$. The vertices $i, j, i\#K, j\#K$ induce a square in Q_n as sketched below.



One immediately observes

$$\mathrm{par}_K(i) = \mathrm{par}_K(j\#K) \quad \text{ and } \quad \mathrm{par}_\ell(i) = \mathrm{par}_\ell(j\#K) + 1.$$

From Lemma 7.3.2(1) and the induction hypothesis we get

$$\frac{\overline{c(f)}}{c(h)} = -\frac{\overline{c(e)}}{c(g)} = -\frac{(-1)^{\mathrm{par}_\ell(j\#K)}\sqrt{t_\ell}}{(-1)^{\mathrm{par}_\ell(i)}\sqrt{t_\ell}} = 1.$$

Consequently, $c(i[i\#K]) = c(f) = \overline{c(h)} = \overline{c([j\#K]j)}$. Inserting this into (7.5) yields $\lambda_i\lambda_j c_{\mathbf{t}}(ij) = c(ij)$ as desired.

Thus, we have proved that the unitary transformation of the weighting c given by the numbers λ_i and λ_j does not change the weighting of the edges $ij \in E_n^{(0)} \cup \cdots \cup E_n^{(K-1)}$. To complete the induction step it only remains to show $c(ij) = \lambda_i\lambda_j c_{\mathbf{t}}(ij)$ for all $ij \in E_n^{(K)}$. For this, assume first $[i]_K = 0$. Then $[j]_K = 1$ and we have

$$\lambda_i\lambda_j c_{\mathbf{t}}(ij) = (-1)^{\mathrm{par}_K(i)} \frac{c(ij)}{|c(ij)|} (-1)^{\mathrm{par}_K(i)} \sqrt{t_K} = c(ij).$$

Otherwise, one has $[i]_K = 1$ and $[j]_K = 0$. It follows again

$$\lambda_i\lambda_j c_{\mathbf{t}}(ij) = (-1)^{\mathrm{par}_K(i)} \frac{c(ij)}{|c(ij)|} (-1)^{\mathrm{par}_K(i)} \sqrt{t_K} = c(ij).$$

□

Theorem 7.5.3. *Let $\rho : C^*(Q_n) \to B(\mathcal{H})$ be an irreducible representation of the hypercube algebra $C^*(Q_n)$ on a Hilbert space $\mathcal{H}$. Then there is a point $\mathbf{t} \in \Delta_{n-1}$ such that ρ is unitarily equivalent to a subrepresentation of the representation $\rho_{\mathbf{t}}$ induced by the weighting $c_{\mathbf{t}}$ from Definition 7.4.1.*

Proof. We prove the statement by induction over n. For $n = 1$ the hypercube Q_1 has two vertices and one edge, $C^*(Q_1) \cong \mathbb{C}$, $\Delta_0 = \{[1]\}$ and one readily checks that $\rho_{[1]}$ is the unique irreducible representation of $C^*(Q_1)$.

For the induction step let $\rho : C^*(Q_n) \to B(\mathcal{H})$ be an irreducible representation. By Theorem 7.2.5, ρ is rank-one, and in view of Lemma 7.3.6(3) we may assume $\rho = \rho_c$ for some admissible weighting $c : E_n \to \mathbb{C}$ of the hypercube Q_n. Further, since ρ is irreducible we know from the same lemma that c is supported on a single sub-hypercube $Q_n(c)$ of Q_n.

First assume $Q_n(c) = Q_n$. Then c is nowhere-vanishing, and by Lemma 7.5.2 there is a $\mathbf{t} \in \Delta_{n-1}$ such that ρ_c is unitarily equivalent to $\rho_{\mathbf{t}}$.

If $Q_n(c) \subsetneq Q_n$, then by Lemma 7.1.6(1) there is some $\ell < n$ such that $Q_n(c) \subset Q_n^{(+\ell)}$ or $Q_n(c) \subset Q_n^{(-\ell)}$, where $Q_n^{(\pm\ell)}$ is the sub-hypercube induced by all vertices $x \in U_n \cup V_n$ with $[x]_\ell = 0$ or $[x]_\ell = 1$ depending on the sign $\pm$. Let us assume $Q_n \subset Q_n^{(+\ell)}$. In this case, ρ_c factors through the surjective map $\pi^{(+\ell)} : C^*(Q_n) \to C^*(Q_{n-1})$ from Notation 7.4.4 since ρ_c vanishes on all p_x with $[x]_\ell = 0$. Thus, there is an irreducible representation $\sigma : C^*(Q_{n-1}) \to B(\mathcal{H})$ with

$$\rho_c = \sigma \circ \pi^{(+\ell)}.$$

By the induction hypothesis there is some $\mathbf{s} \in \Delta_{n-2}$ such that σ is unitarily equivalent to a subrepresentation of $\rho_{\mathbf{s}}$. Consequently, ρ_c is unitarily equivalent to a subrepresentation of $\rho_{\mathbf{s}} \circ \pi^{(+\ell)}$. However, this is again a subrepresentation of $\rho_{\mathbf{t}}$ for $\mathbf{t} = [s_0, \ldots, s_{\ell-1}, 0, s_\ell, \ldots, s_{n-2}] \in \Delta_{n-1}$ by Lemma 7.4.5. □

7.6. Explicit description of hypercube C*-algebras

Finally, we are ready to describe the hypercube C*-algebras $C^*(Q_n)$ explicitly as algebra of matrix-valued continuous functions on the simplex Δ_{n-1}. First, let us introduce a useful notation.

Notation 7.6.1. For $n \in \mathbb{N}$ and $\mathbf{t} = [t_0, \dots, t_{n-1}] \in \Delta_{n-1}$ let

$$Q_n(\mathbf{t}) \subset Q_n$$

be the subgraph obtained by removing all edges $ij \in E_n^{(k)}$ with $t_k = 0$, where $E_n^{(k)} = \{ij \in E_n \mid i = j\#k\}$ as in Notation 7.1.5. Evidently, $Q_n(\mathbf{t})$ partitions Q_n into a disjoint union of sub-hypercubes, where $Q_n(\mathbf{t})$ consists of 2^α-sub-hypercubes of size $2^{n-1-\alpha}$ if $\mathbf{t}$ has α zeros (see the proof of Lemma 7.1.6(1)).

We say that a matrix $A \in M_{U_n}$ is in $\mathbf{t}$-*block diagonal form* if it satisfies

$$A_{i_1 i_2} = 0 \quad \text{whenever } i_1, i_2 \text{ are in distinct connected components of } Q_n(\mathbf{t}).$$

In this case every block of A corresponds to a connected component of $Q_n(\mathbf{t})$.

Example 7.6.2. As an example consider the square Q_2 as sketched in Figure 7.3.5. For $\mathbf{t} = [1, 0] = [t, t'] \in \Delta_1$ the graph $Q_2(\mathbf{t})$ consists of the two horizontal edges. Both connected components contain exactly one vertex from U_2 and thus a matrix $A \in M_{U_2} \cong M_2$ is in $\mathbf{t}$-block diagonal form if and only if it is diagonal.

Lemma 7.6.3. *For every $\mathbf{t} = [t_0, \dots, t_{n-1}] \in \Delta_{n-1}$ and $x \in U_n \cup V_n$ the projection $\rho_{\mathbf{t}}(p_x) \in M_{U_n}$ is in $\mathbf{t}$-block diagonal form where $\rho_{\mathbf{t}} : C^*(Q_n) \to M_{U_n}$ is the representation from Definition 7.4.1.*

Proof. For $i \in U_n$ we have $\rho_{\mathbf{t}}(p_i) = E_{ii}$, which is clearly in $\mathbf{t}$-block diagonal form. For $j \in V_n$ recall that $\rho(p_j)$ is the projection onto the span of the vector $\psi_j \in \mathbb{C}^{U_n}$ given by

$$[\psi_j]_i = \begin{cases} (-1)^{\operatorname{par}_k(i)} \sqrt{t_k}, & \text{if } j = i\#k \text{ for some } k < n, \\ 0, & \text{otherwise.} \end{cases}$$

If $i, i' \in U_n$ are in different connected components of $Q_n(\mathbf{t})$, then there are two possibilities: If j is not a common neighbor of i and i' in Q_n, then $[\psi_j]_i = 0$ or $[\psi_j]_{i'} = 0$. In any event it follows that the (i, i')-th entry of $\rho_{\mathbf{t}}(p_j)$ vanishes. If, on the other hand, j is a common neighbor of i and i', then there are $\ell, k < n$ with $j = i\#\ell$ and $j = i'\#k$. Since i and i' are in different connected components of $Q_n(\mathbf{t})$, we must have $t_k = 0$ or $t_\ell = 0$. Again this entails that the (i, i')-th entry of $\rho_{\mathbf{t}}(p_j)$ vanishes. Thus, $\rho_{\mathbf{t}}(p_j)$ is in $\mathbf{t}$-block diagonal form. $\square$

Theorem 7.6.4. *The C*-algebra $C^*(Q_n)$ is isomorphic to the algebra of continuous functions $f : \Delta_{n-1} \to M_{U_n} \cong M_{2^{n-1}}$ such that $f(\mathbf{t})$ is in $\mathbf{t}$-block diagonal form for all $\mathbf{t} \in \Delta_{n-1}$.*

Proof. For every $\mathbf{t} \in \Delta_{n-1}$ and $x \in U_n \cup V_n$ let $p_x^{\mathbf{t}} := \rho_{\mathbf{t}}(p_x) \in M_{U_n}$. One readily checks that the map $\Delta_{n-1} \to M_{U_n}$ given by $\mathbf{t} \mapsto p_x^{\mathbf{t}}$ is continuous for all $x \in U_n \cup V_n$. Thus, the universal property of $C^*(Q_n)$ yields a $*$-homomorphism

$$\varphi : C^*(Q_n) \to C(\Delta_{n-1}, M_{U_n}), \quad p_x \mapsto (p_x^{\mathbf{t}})_{\mathbf{t} \in \Delta_{n-1}}.$$

It remains to show that φ is injective and that its image consists of all continuous functions $f : \Delta_{n-1} \to M_{U_n}$ such that $f(\mathbf{t})$ is in $\mathbf{t}$-block diagonal form for all $\mathbf{t} \in \Delta_{n-1}$.

To show injectivity, it suffices to note that every irreducible representation ρ of $C^*(Q_n)$ factors through φ. Indeed, by Theorem 7.5.3 every irreducible representation is unitarily equivalent to a subrepresentation of some $\rho_{\mathbf{t}}$ for $\mathbf{t} \in \Delta_{n-1}$.

For surjectivity, first observe that for every $\mathbf{t} \in \Delta_{n-1}$ and $x \in U_n \cup V_n$ the projection $p_x^{\mathbf{t}}$ is in $\mathbf{t}$-block diagonal form by Lemma 7.6.3. On the other hand, whenever $i, i' \in U_n$ are in the same connected component of $Q_n(\mathbf{t})$, then there is a path $i = i_1 j_1 i_2 j_2 \dots j_{m-1} i_m = i'$ in $Q_n(\mathbf{t})$ that connects i and i'. Then one has

$$p_{i_1}^{\mathbf{t}} p_{j_1}^{\mathbf{t}} p_{i_2}^{\mathbf{t}} p_{j_2}^{\mathbf{t}} \dots p_{j_{m-1}}^{\mathbf{t}} p_{i_m}^{\mathbf{t}} \neq 0,$$

and the matrix on the left-hand side has a single non-zero entry at the (i, i')-th position. Thus,

$$\varphi(\Delta_{n-1})|_{\mathbf{t}} := \{\varphi(x)|_{\mathbf{t}} \mid x \in C^*(Q_n)\} \subset M_{U_n}$$

is the algebra of matrices in $\mathbf{t}$-block diagonal form. Now, the Stone-Weierstrass theorem for C*-algebras implies that φ is surjective, see e.g. [28, Theorem 11.1.8]. □

7.7. Application to magic isometries

In this final section we apply our findings about the C^*-algebra of the cube Q_3 in order to solve a problem about magic isometries and magic unitaries. These occur in the context of quantum permutations which go back to the foundational work of Wang [92].

7.7.1. Background on magic isometries and magic unitaries

Motivated by quantum mechanics, Drinfeld and Jimbo introduced quantum groups as a generalization of classical groups in the 1980s [29, 47]. Since then an enormous amount of research has been devoted to the theory of quantum groups which encompasses a wide range of different objects according to additional structure of the group under consideration, e.g. algebraic, discrete, compact, locally compact, etc. We will not try to give an overview of this vast field.

However, the general idea in the definition of quantum groups is always the same: A classical group G is identified with an algebra of functions $\mathrm{Fun}(G)$, and the group structure is translated into an additional structure on this algebra of functions.

In this vein, Woronowisz introduced compact quantum groups as pairs (A, Δ) of a C*-algebra A and a comultiplication $\Delta : A \to A \otimes_{\min} A$ [100, 99, 101]. The first examples of such compact quantum groups were invariably deformations of classical compact groups, e.g. a pivotal example is the q-deformation $\mathrm{SU}_q(2)$ of the special unitary group $\mathrm{SU}(2)$ [102].

An important class of compact quantum groups was later discovered by Wang [92]: He introduced the quantum symmetry group of n points (or quantum permutation group) $S^+(n) = (A, \Delta)$ via a universal property: A is the universal C*-algebra generated by projections u_{ij} for $i, j = 1, \dots, n$ which are the entries of a $n \times n$-matrix $u = (u_{ij}) \in M_n(A)$ satisfying

$$\sum_{i=1}^{n} u_{ij} = 1_A = \sum_{j=1}^{n} u_{ij} \quad \text{for fixed } i, j, \text{ resp.},$$

and Δ is given by $u_{ij} \mapsto \sum_k u_{ik} \otimes u_{kj}$. A matrix u with these properties is called a *magic unitary*. For a survey on quantum permutation groups we refer to [1]. In fact, the group S_n^+ is one from a range of quantum groups which are defined in a similar way. Further, examples are the free unitary and orthogonal groups $\mathrm{U}_n^+, \mathrm{O}_n^+$ introduced by Wang in [90, 91] as well as further free quantum groups, see e.g. [8]. Let us also mention that there is a rich theory on quantum automorphism groups of graphs, see e.g. [81] and the references therein. Excellent references for quantum (permutation) groups are [84, 66, 37].

In what follows, we will not mention quantum groups anymore, but we remind the reader that the problem we solve is motivated by specific questions about compact quantum groups. We will only be concerned with magic unitaries and magic isometries. Let us define these below.

Definition 7.7.1. Let $m \leq n$ and let A be a unital C^*-algebra. An $m \times n$-matrix $P = (P_{ij})_{i \leq m, j \leq n} \in M_{m,n}(A)$ with entries from A is called a *magic isometry* if the entries P_{ij} are projections which form a partition of unity along rows and are pairwise orthogonal along columns, i.e.

$(P_{ij})_{j=1,\dots,n}$ is a partition of unity for each fixed $i \leq m$,

$(P_{ij})_{i=1,\dots,m}$ is a family of pairwise orthogonal projections for each fixed $j \leq n$.

Here, with a partition of unity we mean a family of projections in a unital C^*-algebra that sum up to the unit.

If $m = n$ and the projections P_{ij} form a partition of unity along both rows and columns, then we call P a *magic unitary*.

Banica, Skalski and Sołtan raised the following problem in [7, Section 5], see also Weber's introductory article [98, Problem 3.4]. Related are also Nechita and Pillet's SudoQ games from [65].

Problem 7.7.2. Can every $m \times n$-magic isometry be completed to a magic unitary matrix, i.e. given a magic isometry

$$P = (P_{ij})_{i \leq m, j \leq n} \in M_{m,n}(A),$$

where $A = C^*(P_{ij})$ and $m \leq n$, can one find another C^*-algebra B and a magic unitary

$$U = (U_{ij})_{i,j \leq n} \in M_n(B)$$

such that A embeds into B via $P_{ij} \mapsto U_{ij}$ for $i \leq m, j \leq n$?

In this chapter we consider the special situation where P is a 2×4-magic isometry. To solve Problem 7.7.2 for this case, we will consider two C*-algebras: On the one hand, the universal C*-algebra generated by the entries of a 2×4-magic isometry; on the other hand, the universal C*-algebra generated by the entries of a 4×4-magic unitary. The latter is known as the underlying C*-algebra $C(S_4^+)$ of the quantum permutation group on four points, and it has been extensively studied, see e.g. [5, 6, 3, 2, 48, 49]. As suggested in [7], we will make use of the following faithful representation of $C(S_4^+)$ which is due to Banica and Collins, see also [5].

Theorem 7.7.3 ([4, Theorem 4.1])**.** *Let*

$$c_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; c_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; c_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; c_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be the Pauli matrices in M_2. We identify M_2 with $\mathbb{C}^4$ as Hilbert spaces via the orthonormal basis c_1, c_2, c_3, c_4, where M_2 is equipped with the normalized Hilbert-Schmidt inner product $\langle X, Y \rangle = \frac{1}{2}\mathrm{Tr}(X^*Y)$, and we obtain for every matrix $X \in M_2$ a projection $\Pi(X) \in M_4$ onto $\mathrm{span}(X) \subset M_2$*.*

The $*$-homomorphism $\Phi : C(S_4^+) \to C(\mathrm{SU}(2), M_4)$ given by

$$\Phi : u_{ij} \mapsto (\Pi(c_i x c_j))_{x \in \mathrm{SU}(2)} =: (\sigma_x(u_{ij}))_{x \in \mathrm{SU}(2)} \qquad \text{for } i, j \leq 4,$$

defines a faithful representation of $C(S_4^+)$.

7.7.2. Application of $C^*(Q_3)$ to the magic isometry completion problem

The following is a simple observation which connects 2×4-magic isometries to the hypercube C^*-algebra $C^*(Q_3)$.

Proposition 7.7.4. *The universal C^*-algebra generated by the entries p_{ij} with $i = 1, 2$ and $j = 1, 2, 3, 4$ of a 2×4-magic isometry is isomorphic to the hypercube C*-algebra $C^*(Q_3)$ via*

$$\begin{aligned} p_{11} &\mapsto p_{000}, & p_{21} &\mapsto p_{111}, \\ p_{12} &\mapsto p_{011}, & p_{22} &\mapsto p_{100}, \\ p_{13} &\mapsto p_{110}, & p_{23} &\mapsto p_{001}, \\ p_{14} &\mapsto p_{101}, & p_{24} &\mapsto p_{010}, \end{aligned}$$

where the p_x on the right-hand side are the canonical generators of $C^*(Q_3)$ as in Example 7.4.3.

In what follows we will identify the two C*-algebras from the previous proposition and denote the canonical generators of $C^*(Q_3)$ by p_{ij} with $i = 1, 2$ and $j = 1, 2, 3, 4$.

We can reformulate Problem 7.7.2 in terms of universal C*-algebras as follows.

Lemma 7.7.5. *Problem 7.7.2 has an affirmative answer if the following condition $(*)$ holds: Every irreducible representation of $C^*(Q_3)$ factors through the unique $*$-homomorphism $\varphi : C^*(Q_3) \to C(S_4^+)$ with*

$$\varphi : p_{ij} \mapsto u_{ij} \quad \text{for all } i \leq 2, j \leq 4,$$

where we identify the generators of $C^*(Q_3)$ with the p_{ij} as in Proposition 7.7.4.

This means that whenever $\rho : C^*(Q_3) \to B(\mathcal{H})$ is an irreducible representation, then there exists a representation $\sigma : C(S_4^+) \to B(\mathcal{H})$ such that

$$\rho = \sigma \circ \varphi.$$

Note that the existence and uniqueness of the map φ follows from the universal property of $C(S_4^+)$.

Proof. Let $(P_{ij})_{i=1,2;j=1,2,3,4} \subset M_{2,4}(A)$ be a magic isometry where $A = C^*(P_{ij})$. The elements P_{ij} satisfy the relations (GP1) and (GP2); therefore there is a unique (surjective) $*$-homomorphism $\pi : C^*(Q_3) \to A$ with $\pi(p_{ij}) = P_{ij}$ for $i = 1, 2$ and $j = 1, 2, 3, 4$. Let $I := \ker(\pi)$ be the kernel of π, and let $J \subset C(S_4^+)$ be the closed ideal generated by $\varphi(I)$ with quotient map $\pi_J : C(S_4^+) \to C(S_4^+)/J$. Since $\pi_J \circ \varphi$ vanishes on I, it factors through π as in the diagram below.

$$\begin{tikzcd} C^*(Q_3) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & C(S_4^+) \arrow[d, "\pi_J"] \\ A \arrow[r, dashed, "\psi"] & C(S_4^+)/J \end{tikzcd}$$

Evidently, the magic isometry $(P_{ij})_{i \leq 2; j \leq 4}$ can be filled up to the magic unitary $(\pi_J(u_{ij}))_{i,j \leq 4}$ if the map ψ is injective.

To prove that this is true, it suffices to show that every irreducible representation ρ' of A factors through ψ. But that is implied by $(*)$ as follows. If $\rho' : A \to B(\mathcal{H})$ is irreducible, then the same goes for $\rho := \rho' \circ \pi_I$. By $(*)$ that representation factors as $\rho = \sigma \circ \varphi$ for a suitable irreducible representation $\sigma : C(S_4^+) \to B(\mathcal{H})$. As ρ vanishes on I, σ vanishes on J and therefore factors through π_J as $\sigma = \sigma' \circ \pi_J$. This means that ρ' factors through ψ as $\rho' = \sigma' \circ \psi$. $\square$

To prove that condition $(*)$ in Lemma 7.7.5 is true, we combine our knowledge about $C^*(Q_3)$ with the faithful representations of $C(S_4^+)$ due to Banica and Collins mentioned above. Indeed, all we need is a technical computation.

Lemma 7.7.6. *Let* $\mathbf{t} = [a^2, b^2, c^2] \in \Delta_2$ *with* $a, b, c \geq 0$ *be given and set*

$$x := \begin{pmatrix} is & t+iu \\ -t+iu & -is \end{pmatrix} \in M_2,$$

where

$$s := \frac{a}{\sqrt{2}\sqrt{c+1}}, \qquad t := \frac{b}{\sqrt{2}\sqrt{c+1}}, \qquad u := \sqrt{\frac{c+1}{2}}.$$

Then $x \in \mathrm{SU}(2)$ *and* $\sigma_x \circ \varphi$ *is unitarily equivalent to* $\rho_{\mathbf{t}}$*. Here* σ_x *is the representation of* $C^*(S_4^+)$ *from the previous theorem and* $\rho_{\mathbf{t}}$ *is the representation of* $C^*(Q_3)$ *induced by the weighting* $c_{\mathbf{t}}$ *from Definition 7.4.1.*

Proof. For $x \in \mathrm{SU}(2)$ it suffices to check that $s^2 + t^2 + u^2 = 1$. Indeed,

$$s^2 + t^2 + u^2 = \frac{a^2 + b^2}{2(c+1)} + \frac{c+1}{2} = \frac{a^2 + b^2 + (c+1)^2}{2(c+1)} = \frac{1 + 2c + 1}{2(c+1)} = 1.$$

Next, recall that we identify M_2 with $\mathbb{C}^4$ and that $\Pi(c_i x c_j)$ is the projection onto $\mathrm{span}(c_i x c_j)$. The obtained representation σ_x is unitarily equivalent to σ'_x given by

$$\sigma'_x : u_{ij} \mapsto \Pi(x^* c_i x c_j),$$

for x^* is a unitary transformation. Indeed, if $X \in M_4$ is any unitary matrix, then the map $\mathbb{C}^4 \cong M_2 \ni A \mapsto XA \in M_2 \cong \mathbb{C}^4$ is a unitary transformation. Recall the identification of M_2 and $\mathbb{C}^4$ via the Pauli basis maps

$$x_1 c_1 + x_2 c_2 + x_3 c_3 + x_4 c_4 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Then a calculation confirms

$$x^* c_1 x c_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad x^* c_1 x c_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad x^* c_1 x c_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad x^* c_1 x c_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$x^* c_2 x c_4 = \begin{pmatrix} a \\ ib \\ -ic \\ 0 \end{pmatrix}, \quad x^* c_2 x c_3 = \begin{pmatrix} b \\ -ia \\ 0 \\ ic \end{pmatrix}, \quad x^* c_2 x c_2 = \begin{pmatrix} c \\ 0 \\ ia \\ -ib \end{pmatrix}, \quad x^* c_2 x c_1 = \begin{pmatrix} 0 \\ c \\ b \\ a \end{pmatrix},$$

using

$$c = 2u^2 - 1 = u^2 - s^2 - t^2, \qquad b = 2tu, \qquad a = 2su.$$

By applying the unitary transformation $U = \operatorname{diag}(1, -i, i, -i)$ the vectors in the second row are transformed into

$$\varphi_{24} = \begin{pmatrix} a \\ b \\ c \\ 0 \end{pmatrix}, \qquad \varphi_{23} = \begin{pmatrix} b \\ -a \\ 0 \\ c \end{pmatrix}, \qquad \varphi_{22} = \begin{pmatrix} c \\ 0 \\ -a \\ -b \end{pmatrix}, \qquad \varphi_{21} = \begin{pmatrix} 0 \\ -ic \\ ib \\ -ia \end{pmatrix}.$$

Thus, a comparison with the vectors ψ_j for $j \in U_3$ from Example 7.4.3 immediately yields that $\rho_{\mathbf{t}}$ and σ_x are unitarily equivalent. $\square$

Proposition 7.7.7. *The statement from Problem 7.7.2 is true.*

Proof. By Lemma 7.7.5 it suffices to show that every irreducible representation $\rho : C^*(Q_3) \to B(\mathcal{H})$ factors through φ. However, by Theorem 7.5.3 every irreducible representation of $C^*(Q_3)$ is unitarily equivalent to a subrepresentation of some $\rho_{\mathbf{t}}$ with $\mathbf{t} \in \Delta_2$. Further, by Lemma 7.7.6 every $\rho_{\mathbf{t}}$ factors through φ. $\square$

Part II.

Derived quantum graphs

8. Quantum graphs

In this chapter we collect the basic definitions of quantum sets, quantum graphs and (quantum) isomorphisms between them.

8.1. Quantum sets

In the spirit of Gelfand duality a finite set X is identified with the C*-algebra $C(X)$ of (continuous) functions on X. Then, it becomes natural to regard *any* finite-dimensional C*-algebra B as a non-commutative generalization of a finite set, which is called a *quantum set*. For a number of reasons, it is important to equip the C*-algebra B with a particular functional ψ as well. Then one arrives at the following definition of a quantum set, which is essentially due to Musto, Reutter and Verdon [64].

Definition 8.1.1. A quantum set (B, ψ) is a pair consisting of a finite-dimensional C*-algebra B and a faithful positive functional $\psi : B \to \mathbb{C}$ such that

$$mm^* = \mathrm{Id}_B$$

holds for the multiplication map $m : B \otimes B \to B$ and its adjoint m^* given by $\langle m(a \otimes b), c\rangle_\psi = \langle a \otimes b, m^*(c)\rangle_{\psi\otimes\psi}$ for all $a, b, c \in B$. We call any functional with this property a *quantum set functional* on B.

We think of B as a C*-algebra that is at the same time a Hilbert space with the inner product $\langle \cdot, \cdot\rangle_\psi$ induced by the functional ψ. Maps involving B are generally considered as linear maps with additional properties; thus, we will generally denote the concatenation of two $*$-homomorphisms $\varphi_1, \varphi_2 : B \to B$ by $\varphi_1\varphi_2$ instead of $\varphi_1 \circ \varphi_2$. Elements $b \in B$ are identified with the linear map $\mathbb{C} \to B, \lambda \mapsto \lambda b$, and therefore they have an adjoint

$$b^\dagger : B \to \mathbb{C}, \qquad c \mapsto \langle b, c\rangle_\psi.$$

Remark 8.1.2. An alternative and widely-used definition of a quantum set asks that the functional ψ be a δ-form for some $\delta > 0$ which means that

$$m^*m = \delta^2 \mathrm{Id}_{B\otimes B}$$

holds. The difference between this condition and the above definition is merely a scalar factor: Every quantum set functional is a scaled δ-form and vice versa. See [93] for more details.

The functional ψ is called the *counit* and m^* is called the *comultiplication* on B. Note that m^* is not the comultiplication that is used in the theory of quantum groups; in particular, m^* is generally not a $*$-homomorphism. We collect some of their basic properties below which are well-known, see e.g. [64].

Proposition 8.1.3. *The maps ψ and m^* have the following properties:*

1. *The adjoint ψ^* is the unit in the sense that $\psi^* : \mathbb{C} \to B, \lambda \mapsto \lambda 1_B$.*
2. *The comultiplication is co-associative, i.e.*
$$(\mathrm{Id}_B \otimes m^*)m^* = (m^* \otimes \mathrm{Id}_B)m^*.$$
3. *The comultiplication satisfies*
$$(\psi \otimes \mathrm{Id}_B)m^* = \mathrm{Id}_B = (\mathrm{Id}_B \otimes \psi)m^*.$$

Example 8.1.4.

1. Every classical finite set X gives rise to a quantum set $(C(X), \psi_X)$, where $\psi_X : C(X) \to \mathbb{C}$ is the functional given by $\psi_X(f) = \sum_{x\in X} f(x)$ for all $f \in C(X)$. Indeed, in this case $m^*(e_x) = e_x \otimes e_x$ holds for all $x \in X$, and thus $mm^*(e_x) = m(e_x \otimes e_x) = e_x$. Here e_x denotes the indicator function at $x \in X$.
2. For every matrix algebra M_n with $n \in \mathbb{N}$, a quantum set is obtained by equipping M_n with the functional $\psi := n\mathrm{Tr}$, where Tr is the un-normalized trace on M_n. In fact, this is the unique *tracial* quantum set functional on M_n. In this case, the comultiplication is given by
$$m^*(E_{ij}) = \frac{1}{n}\sum_{k=1}^{n} E_{ik} \otimes E_{kj}.$$

8.2. Quantum graphs

A simple (directed) graph Γ on a classical finite set $\{1, \dots, n\}$ can be described in different ways. First, there is the adjacency matrix
$$A = (A_{ij})_{i,j\leq n} \in M_n,$$
where $A_{ij} = 1$ if and only if the graph Γ has an edge from i to j. Next, one can describe Γ by its edge set $E_\Gamma \subset \{(i,j) \mid i,j \leq n\}$ which is equivalently given by a projection
$$P = \sum_{i\to j} e_i \otimes e_j \in \mathbb{C}^n \otimes \mathbb{C}^n,$$
where $i \to j$ denotes the relation "there is an edge from i to j in Γ". Finally, one can describe Γ by the space
$$V = \mathrm{span}\{E_{ij} \mid \Gamma \text{ has an edge from } i \text{ to } j\} \subset M_n.$$
Quantum graphs can be defined by generalizing all these three descriptions to the setting of quantum sets.

Definition 8.2.1. Let (B, ψ) be a quantum set. A quantum graph is given by either of the following:

1. A linear operator $A : B \to B$ which satisfies

$$m(A \otimes A)m^* = A \quad \text{and} \quad A(b^*) = A(b)^* \quad \text{for all } b \in B.$$

   In this case, we call A a *quantum adjacency matrix* on the quantum set (B, ψ).

2. A projection $P \in B \otimes B^{\mathrm{op}}$ where B^{op} is the opposite C*-algebra of B. We call P a *quantum edge projection* on the quantum set (B, ψ).

3. A subspace $V \subset B(\mathcal{H})$ which is a B'–B'–bimodule where $B \subset B(\mathcal{H})$ is an arbitrary embedding of B and B' is the commutant of B in $B(\mathcal{H})$. We call V a *quantum edge space* on the quantum set (B, ψ).

We also call such a quantum graph a directed quantum graph to emphasize that we do not require additional properties which make the quantum graph undirected.

Remark 8.2.2. The operation $m(\cdot \otimes \cdot)m^*$ generalizes the entrywise (or Schur) product of two matrices in the classical case. Therefore, a map A satisfying $m(A \otimes A)m^*$ is also called Schur-idempotent. Matsuda showed that the second condition $A(b^*) = A(b)^*$ is equivalent to asking that A is completely positive if A is Schur-idempotent [62].

Historically, the first definitions of quantum graphs by Duan–Severini–Winter and Weaver [30, 97] were given in terms of quantum edge spaces. Weaver also used the quantum edge projection to prove that his definition does not depend on the particular embedding $B \subset B(\mathcal{H})$. The quantum adjacency matrix was introduced by Musto–Reutter–Verdon as part of a categorical framework for quantum sets and quantum functions [64]. In the literature, it is common to restrict attention to quantum graphs on tracial quantum sets or to undirected quantum graphs. Above we presented the most general definition of quantum graphs, and we will briefly discuss particular properties of quantum graphs later in this section.

The equivalences between the different definitions of quantum graphs have been proved in different generality in the literature. Things are most clear in the case of tracial quantum sets; if the quantum set is non-tracial there are indeed slightly different approaches from Wasilewski and Daws to prove equivalences between quantum adjacency matrices, quantum edge projections and quantum edge spaces [93, 27]. We will discuss the general equivalences in Wasilewski's framework in detail for quantum sets of the form (M_n, ψ) in a later section. For now, we collect the classical equivalences for tracial quantum graphs.

We use the following notation: If $T = \sum_{i \leq n} A_i \otimes B_i \in B \otimes B^{\mathrm{op}}$, $B \subset B(\mathcal{H})$ and $S \in B(\mathcal{H})$, then we write

$$T \# S := \sum_{i \leq n} A_i S B_i,$$

where A_i and B_i are identified with their images in $B(\mathcal{H})$. Note that B^{op} and B coincide as vector spaces, and thus every $B_i \in B^{\mathrm{op}}$ is first regarded as an element of B and then identified with its image under the embedding $B \subset B(\mathcal{H})$. We require the B_i to be in the opposite C*-algebra precisely because they are multiplied from the right in the definition of $T\#S$.

Proposition 8.2.3. *Let (B,ψ) be a tracial quantum set.*

1. *There is a 1-1-correspondence between quantum adjacency matrices $A : B \to B$ and quantum edge projections $P \in B \otimes B^{\mathrm{op}}$ given by*

$$\begin{aligned} A &\mapsto P_A := (A \otimes \mathrm{Id}_B)m^*(1_B), \\ P &\mapsto A_P := (\mathrm{Id}_B \otimes \psi m)(P \otimes \mathrm{Id}_B). \end{aligned}$$

2. *Given an embedding $B \subset B(\mathcal{H})$, there is a 1-1-correspondence between quantum edge projections $P \in B \otimes B^{\mathrm{op}}$ and quantum edge spaces $V \subset B(\mathcal{H})$ given by*

$$\begin{aligned} P &\mapsto V_P := \{P\#T \mid T \in B(\mathcal{H})\} \\ V &\mapsto P_V, \end{aligned}$$

   where in the last line P_V is the unique projection such that the left ideal

$$\{S \in B \otimes B^{\mathrm{op}} \mid S\#V = \{0\}\} \subset B \otimes B^{\mathrm{op}}$$

   is of the form $(B \otimes B^{\mathrm{op}})(1 - P_V)$.

The first equivalence is due to Musto, Reutter and Verdon [64], and the second equivalence is due to Weaver [97]. We only sketch the ideas of the proof. In fact, the second equivalence does not involve the functional ψ at all, and thus it holds for any quantum set.

Proof. Ad (1). One can show that the map $A \mapsto P_A$ identifies the Schur-product $m(\cdot \otimes \cdot)m^*$ on the left-hand side with the standard product in $B \otimes B^{\mathrm{op}}$ on the right-hand side. Thus, A is Schur-idempotent if and only if P_A is idempotent in $B \otimes B^{\mathrm{op}}$. Further, the condition $A(b^*) = A(b)^*$ for quantum adjacency matrices is equivalent to asking that P_A is self-adjoint in $B \otimes B^{\mathrm{op}}$. Hence, A is a quantum adjacency matrix if and only if P_A is a quantum edge projection. One easily checks that $P \mapsto A_P$ is inverse of $A \mapsto P_A$.

Ad (2). Since B is finite-dimensional, one can essentially assume that $\mathcal{H}$ is finite-dimensional as well (see [97] for details). Then, there is a 1-1-correspondence between left ideals $\mathcal{I}$ of $B \otimes B^{\mathrm{op}}$ and quantum edge spaces $V \subset B(\mathcal{H})$, which is given by

$$\begin{aligned} \mathcal{I} &\mapsto V_{\mathcal{I}} := \{S \in B(\mathcal{H}) \mid \mathcal{I}\#S = \{0\}\}, \\ V &\mapsto \mathcal{I}_V := \{T \in B \otimes B^{\mathrm{op}} \mid T\#V = \{0\}\}. \end{aligned}$$

Since $\mathcal{H}$ is finite-dimensional, this traces back to the fact that every left ideal in M_k is of the form M_kQ for some projection Q, and hence there is a 1-1-correspondence between left ideals in M_k and the subspaces that they annihilate. By that same argument left ideals in $B \otimes B^{\mathrm{op}}$ correspond to projections $P \in B \otimes B^{\mathrm{op}}$. Since the map $\mathcal{I} \mapsto V_{\mathcal{I}}$ is order-reversing it is suitable to choose for given $V \subset B(\mathcal{H})$ the projection P_V such that $\mathcal{I}_V = (B \otimes B^{\mathrm{op}})(1 - P_V)$. $\square$

Example 8.2.4. 1. Recall the quantum set $(C(X), \psi_X)$ associated with a classical finite set X from Example 8.1.4. Quantum graphs on $(C(X), \psi_X)$ are exactly the classical graphs on X. Indeed, the operation $m(\cdot \otimes \cdot)m^*$ produces the entrywise product of two matrices in this case, and thus a map/matrix $A : \mathbb{C}^n \to \mathbb{C}^n$ is Schur-idempotent if and only if its entries are 0 or 1. Further, any projection $P \in C(X) \otimes C(X)^{\mathrm{op}}$ is uniquely determined by a set of pairs $(i,j) \in X \times X$ which corresponds to a set of directed edges on X. Finally, given a unital inclusion $C(X) \subset B(\mathcal{H})$, a space $V \subset B(\mathcal{H})$ is a B'–B'–bimodule if and only if it is of the form

$$V = \{E_{ij} \otimes T \mid (i,j) \in E_\Gamma, T \in M_k\} \subset B(\mathcal{H})$$

for some $E_\Gamma \subset \{(i,j) \mid i,j \in X\}$, where $\mathcal{H} = C(X) \otimes \mathbb{C}^k$ and $M_k = B(\mathbb{C}^k)$ denotes the algebra of $k \times k$-matrices.

2. Given an arbitrary quantum set (B, ψ), the identity map Id_B is a quantum adjacency matrix. Classically, this yields the graph that has a loop at every vertex and no other edges. If (B, ψ) is tracial, then the associated quantum edge space is $\mathrm{span}(\mathrm{Id}_{\mathcal{H}})$.

3. On an arbitrary quantum set (B, ψ), the linear map $\psi(\cdot)1_B$ is a quantum adjacency matrix. Classically, this construction yields the complete graph on the underlying vertex set, and in general the associated quantum graph is regarded as the complete quantum graph on (B, ψ). If (B, ψ) is tracial, then this quantum graph has the full quantum edge space $B(\mathcal{H})$.

Finally, let us collect some important properties of quantum graphs. We describe all properties in terms of properties of the quantum adjacency matrix, but they can be formulated in terms of the quantum edge projection and the quantum edge space as well.

Definition 8.2.5. Let (B, ψ) be a quantum set and let $A : B \to B$ be a quantum adjacency matrix on (B, ψ). We say that the quantum graph associated to A is

1. *tracial* if ψ is a tracial functional,
2. *loopfree* if $m(A \otimes \mathrm{Id}_B)m^* = 0$,
3. *reflexive* if $m(A \otimes \mathrm{Id}_B)m^* = \mathrm{Id}_B$,
4. *undirected* if $A = A^*$,
5. *d-regular* for some $d \geq 0$ if $A(1) = d$.

Let us remark that for non-tracial quantum graphs there is an alternative notion of KMS-undirectedness, where one asks that A is equal to its adjoint with respect to the KMS inner product on B instead of the GNS inner product [93].

8.3. Quantum isomorphisms of quantum graphs

In this section we recall the definition of isomorphisms and quantum isomorphisms between quantum sets and quantum graphs. We mainly follow the theory of Musto, Reutter and Verdon [64] with the difference that we do not take a categorical perspective on quantum sets and quantum graphs but work with explicit C*-algebras. In particular, throughout this section we consider only quantum adjacency matrices and do not speak about the other equivalent formulations of quantum graphs.

Definition 8.3.1. Let (B_1, ψ_1) and (B_2, ψ_2) be quantum sets and let $A_1 : B_1 \to B_1$ and $A_2 : B_2 \to B_2$ be quantum graphs on (B_1, ψ_1) and (B_2, ψ_2), respectively.

1. The two quantum sets are *isomorphic* if there is a $*$-isomorphism $\varphi : B_1 \to B_2$ such that $\psi_2 \varphi = \psi_1$. We denote by $\mathrm{Aut}(B, \psi)$ the group of quantum set automorphisms of a quantum set (B, ψ).

2. The two quantum graphs are *isomorphic* if there is an isomorphism $\varphi : B_1 \to B_2$ of the underlying quantum sets such that $\varphi A_1 = A_2 \varphi$. We denote by $\mathrm{Aut}(A)$ the group of quantum graph automorphisms of a quantum graph A.

3. The two quantum sets are *quantum isomorphic*, written $(B_1, \psi_1) \cong_q (B_2, \psi_2)$, if there is a finite-dimensional Hilbert space $\mathcal{H}$ and a unital $*$-homomorphism

$$\rho : B_1 \to B_2 \otimes B(\mathcal{H}),$$

such that the map

$$p : L^2(B_1) \otimes \mathcal{H} \to L^2(B_2) \otimes \mathcal{H}, \quad a \otimes \xi \mapsto \rho(a)(1 \otimes \xi)$$

is unitary as an operator between Hilbert spaces.

4. The two quantum graphs are *quantum isomorphic*, written $A_1 \cong_q A_2$, if there is a quantum isomorphism $\rho : B_1 \to B_2 \otimes B(\mathcal{H})$ between the underlying quantum sets such that

$$\rho A_1 = (A_2 \otimes \mathrm{Id}_{B(\mathcal{H})})\rho$$

holds.

Remark 8.3.2. The previous definition is a rephrased version of the definition given by Musto, Reutter and Verdon in [64]. They define a quantum function from (B_1, ψ_1) to (B_2, ψ_2) as a linear map

$$P : \mathcal{H} \otimes B_1 \to B_2 \otimes \mathcal{H},$$

which satisfies certain properties. Their map P corresponds to our map p after swapping the tensor legs in the domain, and their properties on P correspond to asking that the associated map ρ is a $*$-homomorphism.

Moreover, Musto, Reutter and Verdon have additional properties which P must satisfy in order to be called *bijective*. In [64, Theorem 4.8] several equivalent characterizations of bijective quantum functions are given. One characterization asks that the map P is unitary, and that is the property that we use in our definition of quantum isomorphisms.

Let us finally mention that the above formulation of quantum isomorphisms follows mainly Brannan et al. [15] with the only difference that we adopt a basis-free perspective.

Notation 8.3.3. It is convenient to treat a quantum isomorphism $\rho : B_1 \to B_2 \otimes B(\mathcal{H})$ as if it were a map from B_1 to B_2. For that we use the following notational conventions.

1. If $T : B_2 \to B_3$ is a linear map, then we write

$$T\rho \text{ for the map } (T \otimes \mathrm{Id}_{\mathcal{H}})\rho : B_1 \to B_3 \otimes B(\mathcal{H}).$$

   An equation of the form

$$T\rho = S$$

   for some $S : B_1 \to B_3$ is, accordingly, to be understood as

$$(T \otimes \mathrm{Id}_{\mathcal{H}})\rho = S \otimes \mathrm{Id}_{\mathcal{H}}.$$

2. Further, we introduce a modified tensor product $\rho \boxtimes \rho$. Indeed, for any $x, y \in B_1$ the elements $\rho(x)$ and $\rho(y)$ can be interpreted as matrices with values in B_2 since $\mathcal{H}$ is finite-dimensional. Let us embed the entries of $\rho(x)$ in the left tensor leg of $B_2 \otimes B_2$, and the entries of $\rho(y)$ in the right tensor leg of $B_2 \otimes B_2$. Then, $\rho(x)$ and $\rho(y)$ are two matrices with values in $B_2 \otimes B_2$. These two matrices can be multiplied according to the usual matrix multiplication rules. Then we obtain a new matrix of the same size as before with entries in $B_2 \otimes B_2$. It is this matrix that we denote by $(\rho \boxtimes \rho)(x \otimes y)$. More precisely, we write

$$\rho \boxtimes \rho \text{ for the map } (\mathrm{Id}_{B_2} \otimes \mathrm{Id}_{B_2} \otimes m_{B(\mathcal{H})})\Sigma_{1,3,2,4}(\rho \otimes \rho),$$

   where $\Sigma : B_2 \otimes B(\mathcal{H}) \otimes B_2 \otimes B(\mathcal{H}) \to B_2 \otimes B_2 \otimes B(\mathcal{H}) \otimes B(\mathcal{H})$ swaps the two tensor factors in the middle and $m_{B(\mathcal{H})} : B(\mathcal{H}) \otimes B(\mathcal{H}) \to B(\mathcal{H})$ is the multiplication map on $B(\mathcal{H})$.

3. Accordingly, if $S : B_2 \to B_3$ and $T : B_2 \to B_4$ are linear maps, then we write

$$\begin{aligned}&(S \otimes T)(\rho \boxtimes \rho) \text{ for the map} \\ &\qquad (S \otimes T \otimes \mathrm{Id}_{\mathcal{H}})(\mathrm{Id}_{B_2} \otimes \mathrm{Id}_{B_2} \otimes m_{B(\mathcal{H})})\Sigma_{1,3,2,4}(\rho \otimes \rho).\end{aligned}$$

Let us stress that these notational conventions are in accordance with the theory of Musto, Reutter and Verdon for computing with quantum functions and quantum isomorphisms.

The following proposition collects some elementary properties of quantum isomorphisms. In fact, these properties can be used for defining quantum isomorphisms as well, see [64, Theorem 4.8].

Proposition 8.3.4. *Let $\rho : B_1 \to B_2 \otimes B(\mathcal{H})$ be a quantum isomorphism between the quantum sets (B_1, ψ_1) and (B_2, ψ_2). Then the following properties hold:*

1. $m_2(\rho \boxtimes \rho) = \rho m_1$,
2. $\psi_2 \rho = \psi_1$,
3. $(\rho \boxtimes \rho) m_1^* = m_2^* \rho$,
4. $\rho \psi_1^* = \psi_2^*$,

where m_1 (m_2) is the multiplication map on B_1 (B_2).

Proof. The first equation says that ρ is multiplicative, and this must be true since ρ is a $*$-homomorphism. The second equation follows from the fact that p is unitary, see [15, Lemma 4.2]. The last two equations are essentially the properties that Musto, Reutter and Verdon require for a quantum function to be bijective, see [64, Theorem 4.8]. $\square$

9. Quantum graphs on M_2

In this section we recall the classification of loopfree undirected quantum graphs on $(M_2, 2\mathrm{Tr})$ obtained by Matsuda [62] and Gromada [41]. Further, we present a more general classification of loopfree directed quantum graphs on $(M_2, 2\mathrm{Tr})$. This is only a special case of an extensive classification of directed quantum graphs on any quantum set of the form (M_2, ψ) that was obtained by the author and Nina Kiefer in [50].

Recall that an undirected quantum graph is given by a quantum adjacency matrix that is self-adjoint with respect to the GNS-inner product on B. A directed quantum graph is, on the other hand, given by an arbitrary quantum adjacency matrix.

9.1. Equivalent definitions of quantum graphs on M_2

To start with, let us spell out the equivalences between quantum adjacency matrices, quantum edge projections and quantum edge spaces on the quantum set $(M_2, 2\mathrm{Tr})$ in detail. Throughout this section we consider M_2 as a Hilbert space with the GNS inner product given by

$$\langle x, y\rangle = 2\mathrm{Tr}(x^* y).$$

Sometimes we also need to consider M_2 with the inner product induced by the normalized trace $\mathrm{tr} = \frac{1}{2}\mathrm{Tr}$.

The following propositions give concrete formulas for translating between quantum adjacency matrices, quantum edge projections and quantum edge spaces on $(M_2, 2\mathrm{Tr})$. They follow closely the more general exposition of Wasilewski [93, Section 3], and we refer to his paper for the proofs.

Proposition 9.1.1 ($V \mapsto A$, [93, Proposition 3.30])**.** *Let $(X_i)_i$ be an orthonormal basis of the quantum edge space $V \subset M_2$ with respect to the (GNS)-inner product induced by the* normalized *trace* tr *on M_2. Then, the map*

$$A : M_2 \to M_2, \quad x \mapsto \sum_i X_i x X_i^*$$

is a quantum adjacency matrix, and the assignment $V \mapsto A$ gives a well-defined[1] 1-1-correspondence between quantum edge spaces and quantum adjacency matrices.

Remark 9.1.2. The linear spaces M_2 and $\mathbb{C}^4$ are naturally identified via the map $\mathrm{vec} : M_2 \ni E_{ij} \mapsto e_{(i-1)2+j} \in \mathbb{C}^4$ where the e_i with $i = 1, \dots, 4$ are the standard

[1] This means that A does only depend on the space V and not on the choice of orthonormal basis.

generators of $\mathbb{C}^4$. Then, a linear map on M_2 is identified with a linear map on $\mathbb{C}^4$ which in turn is nothing but a 4×4-matrix. Generally, one has for any $R, S, X \in M_2$

$$\mathrm{vec}(RXS) = (R \otimes S^t)\mathrm{vec}(X),$$

see e.g. [95, Section 1.1.2]. Here S^t is the transpose of S and $\otimes$ denotes the Kronecker product given by

$$[R \otimes S]_{(i-1)n+j,(k-1)n+\ell} = R_{ik} S_{j\ell}.$$

Thus, we can identify the adjacency matrix $A : M_2 \to M_2$ from the previous proposition with the 4×4-matrix

$$A := \sum_i X_i \otimes (X_i^*)^t = \sum_i X_i \otimes \overline{X_i}.$$

Proposition 9.1.3 ($V \mapsto P$, [93, Proposition 3.29])**.** *Let $(X_i)_i$ be an orthonormal basis of the quantum edge space $V \subset M_n$ with respect to the KMS inner product induced by ψ^{-1} and define $P \in M_n \otimes M_n^{\mathrm{op}}$ as*

$$P = \sum_{i,k,l} X_i e_{kl} \otimes X_i^* e_{lk},$$

where $X_i^* e_{lk}$ is regarded as an element of M_n^{op}. Then P is a quantum edge projection, and the assignment $V \mapsto P$ gives a well-defined 1:1-correspondence between quantum edge spaces and quantum edge projections.

9.2. Pauli matrices and quantum edge spaces on M_2

As first observed by Gromada, it is most convenient to classify quantum graphs on $(M_2, 2\mathrm{Tr})$ in terms of their quantum edge spaces $V \subset M_2$. It is not hard to see that two quantum graphs on $(M_2, 2\mathrm{Tr})$ with quantum edge spaces V_1, V_2 are isomorphic if and only if there is an automorphism of M_2 mapping V_1 to V_2. Thus, for a classification of quantum graphs on $(M_2, 2\mathrm{Tr})$ one needs to understand the automorphism group of M_2 and its action on the subspaces of M_2. In this context it is useful to write the elements of M_2 as linear combinations of the identity matrix I_2 and the Pauli matrices $\sigma_1, \sigma_2, \sigma_3 \in M_2$. Let us collect the most important facts in the following proposition. We do not give a proof since everything is well-known.

Proposition 9.2.1. *Let*

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

be the Pauli matrices and write $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)^t$. Then, the set $\{I_2, \sigma_1, \sigma_2, \sigma_3\}$ is an orthogonal basis of M_2 and every matrix $X \in M_2$ can be written uniquely as

$$X = x_0 I_2 + \mathbf{x}^t \boldsymbol{\sigma}$$

where $x_0 \in \mathbb{C}$ *and* $\mathbf{x}^t = (x_1, x_2, x_3) \in \mathbb{C}^3$. *Further,* X *is orthogonal to* I_2 *if and only if* $x_0 = 0$, *and* X *is self-adjoint if and only if* $x_0, x_1, x_2, x_3 \in \mathbb{R}$. *Every automorphism of* M_2 *is of the form*

$$M_2 \ni x_0 I_2 + \mathbf{x}\boldsymbol{\sigma}^t \mapsto x_0 I_2 + (R\mathbf{x})\boldsymbol{\sigma}^t \in M_2,$$

for some rotation $R \in \mathrm{SO}(3)$.

A quantum graph on $(M_2, 2\mathrm{Tr})$ is loopfree if and only if it is orthogonal to I_2, i.e. if and only if its quantum edge space is of the form

$$V := \{\mathbf{x}\boldsymbol{\sigma}^t \mid \mathbf{x} \in S\}$$

for some subspace $S \subset \mathbb{C}^3$. To simplify the following discussions let us call S the *Pauli edge space* of the quantum graph. If the quantum graph is undirected, then the Pauli edge space S is completely determined by the real subspace of S given by

$$\tilde{S} := S \cap \mathbb{R}^3.$$

Thus, we arrive at the following proposition which is fundamental for classifying loopfree quantum graphs on $(M_2, 2\mathrm{Tr})$.

Proposition 9.2.2. *Two loopfree quantum graphs on* $(M_2, 2\mathrm{Tr})$ *with Pauli edge spaces* $S_1, S_2 \subset \mathbb{C}^3$ *are isomorphic if and only if there is a rotation* $R \in \mathrm{SO}(3)$ *such that*

$$S_1 = RS_2.$$

If both quantum graphs are undirected, then they are even isomorphic if and only if there is a rotation $R \in \mathrm{SO}(3)$ *such that*

$$\tilde{S}_1 = R\tilde{S}_2.$$

9.3. Classification of loopfree quantum graphs on $(M_2, 2\mathrm{Tr})$

Independently of each other, Matsuda [62] and Gromada [41] classified all undirected, loopfree quantum graphs on $(M_2, 2\mathrm{Tr})$. Let us recall here the statement together with Gromada's simple proof.

Theorem 9.3.1 ([41, 62]). *There are exactly four non-isomorphic loopfree undirected quantum graphs on* $(M_2, 2\mathrm{Tr})$ *which are completely determined by the dimension of their quantum edge space* $V \subset M_2$. *Up to isomorphism, these quantum graphs are given by*

Figure 9.1.: Classical graphs on four vertices which are quantum isomorphic to quantum graphs on $(M_2, 2\mathrm{Tr})$.

the following quantum adjacency matrices:

$$
\begin{aligned}
A_0 &= 0, \\
A_1 &= \sigma_1 \otimes \overline{\sigma_1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\
A_2 &= \sigma_1 \otimes \overline{\sigma_1} + \sigma_2 \otimes \overline{\sigma_2} = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}, \\
A_3 &= \sigma_1 \otimes \overline{\sigma_1} + \sigma_2 \otimes \overline{\sigma_2} + \sigma_3 \otimes \overline{\sigma_3} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

Here, we write a map from M_2 into M_2 as a 4×4-matrix with respect to the basis $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ of M_2 in line with Remark 9.1.2.

Further, the quantum graph A_d is d-regular for $d = 0, 1, 2, 3$.

Proof. In view of Proposition 9.2.2 it suffices to observe that any two real subspaces $\tilde{S}_1, \tilde{S}_2 \subset \mathbb{R}^3$ of the same dimension are related by a rotation $R \in \mathrm{SO}(3)$. The adjacency matrices are then obtained according to Proposition 9.1.1 using that $\{I_2, \sigma_1, \sigma_2, \sigma_3\}$ is an orthonormal basis of M_2 with respect to the GNS inner product induced by tr. It is not hard to check the regularity of the obtained quantum graphs. $\square$

Before we move on to loopfree directed quantum graphs on $(M_2, 2\mathrm{Tr})$ let us also recall a remarkable discovery of Matsuda: All four loopfree undirected quantum graphs on $(M_2, 2\mathrm{Tr})$ are quantum isomorphic to classical graphs. We will later describe this as a special case of a more general phenomenon.

Theorem 9.3.2 ([62, Corollary 4.9])**.** *The four non-isomorphic loopfree undirected quantum graphs on $(M_2, 2\mathrm{Tr})$ are quantum isomorphic to the classical graphs on four vertices depicted in Figure 9.3.*

To classify loopfree directed quantum graphs on $(M_2, 2\mathrm{Tr})$ one needs to determine when two *complex* subspaces $S_1, S_2 \subset \mathbb{C}^3$ are related by a rotation $R \in \mathrm{SO}(3)$ via

$S_1 = RS_2$. In fact, it completely suffices to solve this problem for one-dimensional subspaces S_1, S_2 since there is only one subspace of dimension 0 and 3, respectively, and every subspace of dimension 2 is the orthogonal complement of a one-dimensional subspace.

The task of classifying one-dimensional subspaces of $\mathbb{C}^3$ up to rotations from $\mathrm{SO}(3)$ essentially boils down to classifying points on the Riemann sphere up to rotations. The decisive lemma is the following.

Lemma 9.3.3. *For every one-dimensional subspace $S \subset \mathbb{C}^2$ there is some $R \in \mathrm{O}(2)$ and a unique $\beta \in [0,1]$ such that*

$$S = R \operatorname{span}\left\{\begin{pmatrix} 1 \\ i\beta \end{pmatrix}\right\}$$

holds.

Proof. The central tool for this is the Möbius transformation which defines an action of $\mathrm{GL}_2(\mathbb{C})$ on $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

We will use the following two facts:

1. To any nonzero vector $u = \begin{pmatrix} z_1 & z_2 \end{pmatrix}^t \in \mathbb{C}^2$ we may associate the number

$$\varphi(u) := \frac{z_1}{z_2}$$

in the complex sphere $\hat{\mathbb{C}}$. This number completely describes the line $\operatorname{span}\{u\}$, and for any two nonzero vectors $u_1, u_2 \in \mathbb{C}^2$ one has

$$u_1 \in \operatorname{span}\{u_2\} \quad \Leftrightarrow \quad \varphi(u_1) = \varphi(u_2).$$

2. For any matrix $X \in \mathrm{GL}_2(\mathbb{C})$ and vector $u \in \mathbb{C}^2$ one has

$$\varphi(Xu) = X \cdot \varphi(u).$$

Let $S = \operatorname{span}(v)$ for some non-zero vector $v \in \mathbb{C}^2$. Then, we have

$$Rv \in \operatorname{span}\{v_\beta\} \quad \Leftrightarrow \quad R \cdot \varphi(v) = \varphi(v_\beta)$$

for all $R \in \mathrm{O}(2)$ and $v_\beta := \begin{pmatrix} 1 & i\beta \end{pmatrix}^t \in \mathbb{C}^2$. Hence, to prove the statement we need to consider the orbits of the action of $\mathrm{O}(2)$ on $\hat{\mathbb{C}}$ given by the Möbius transformation.

First let us consider only the action by $\mathrm{SO}(2) \subset \mathrm{O}(2)$. Any special orthogonal matrix $R \in \mathrm{SO}(2)$ is of the form

$$R = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

for some $a + ib \in S^1$ with $a, b \in \mathbb{R}$. For the matrix

$$P := \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

one has

$$P \begin{pmatrix} a & b \\ -b & a \end{pmatrix} P^{-1} = \begin{pmatrix} a + ib & 0 \\ 0 & a - ib \end{pmatrix}.$$

Writing $a + ib = e^{it}$, it follows for all $z \in \hat{\mathbb{C}}$

$$\left(P \begin{pmatrix} a & b \\ -b & a \end{pmatrix} P^{-1} \right) \cdot z = \frac{e^{it} z}{e^{-it}} = e^{2it} z.$$

Using that, we get for any two $z_1, z_2 \in \hat{\mathbb{C}}$

$$\begin{aligned} & R \cdot z_1 = z_2 \quad \text{for some } R \in \mathrm{SO}(2) \\ \Leftrightarrow \quad & (PRP^{-1}) \cdot (P \cdot z_1) = P \cdot z_2 \quad \text{for some } R \in \mathrm{SO}(2) \\ \Leftrightarrow \quad & e^{2it}(P \cdot z_1) = P \cdot z_2 \quad \text{for some } t \in \mathbb{R} \\ \Leftrightarrow \quad & |P \cdot z_1| = |P \cdot z_2|, \end{aligned}$$

where we use the convention $|\infty| = \infty$. Thus, the orbits of $\mathrm{SO}(2) \curvearrowright \hat{\mathbb{C}}$ are exactly

$$O_r = \{P^{-1} \cdot z : |z| = r\} \quad \text{with } r \in [0, \infty].$$

A system of representatives is given by the numbers $-i\beta^{-1}$ with $\beta \in [-1, 1]$. Indeed, we have

$$-i\beta^{-1} = P^{-1} \cdot (P \cdot (-i\beta^{-1})) \quad \text{with} \quad P \cdot (-i\beta^{-1}) = \frac{-i\beta^{-1} - i}{-i\beta^{-1} + i} = \frac{1 + \beta}{1 - \beta},$$

and the latter term defines a bijective map from $[-1, 1]$ to $[0, \infty]$. Since for the vector $v_\beta := \begin{pmatrix} 1 & i\beta \end{pmatrix}^t \in \mathbb{C}^2$ we have $\varphi(v_\beta) = \frac{1}{i\beta} = -i\beta^{-1}$, it follows that

$$S = Rv_\beta$$

for some $R \in \mathrm{SO}(2)$ and unique $\beta \in [-1, 1]$.

It remains to consider the more general action of O(2). Any matrix in O(2) is of the form

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^{\varepsilon} R \quad \text{with} \quad \varepsilon \in \{0,1\}, R \in \mathrm{SO}(2).$$

In order to classify the orbits of the action $\mathrm{O}(2) \curvearrowright \hat{\mathbb{C}}$ given by the Möbius transformation we, therefore, need only find out which orbits O_r of $\mathrm{SO}(2) \curvearrowright \hat{\mathbb{C}}$ are linked by the matrix $\mathrm{diag}(-1,1)$. For any $z \in \hat{\mathbb{C}}$ one has

$$\left(P \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} P^{-1} \right) \cdot z = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \cdot z = \frac{-1}{-z} = \frac{1}{z}.$$

Hence, for any $z_1, z_2 \in \hat{\mathbb{C}}$ we have

$$\begin{aligned} & \mathrm{diag}(-1,1) \cdot z_1 = z_2 \\ \Leftrightarrow \quad & \left(P \mathrm{diag}(-1,1) P^{-1} \right) \cdot (P \cdot z_1) = P \cdot z_2 \\ \Leftrightarrow \quad & (P \cdot z_1)^{-1} = P \cdot z_2, \end{aligned}$$

and for any $z \in \hat{\mathbb{C}}$ the numbers $P^{-1} \cdot z$ and $P^{-1} \cdot z^{-1}$ are in the same orbit of $\mathrm{O}(2) \curvearrowright \hat{\mathbb{C}}$. Consequently, the orbits of this action are exactly

$$O'_r := \{P^{-1} \cdot z : |z| \in \{r, r^{-1}\}\} \quad \text{with } r \in [1, \infty].$$

A system of representatives is given by $\{-i\beta^{-1} : \beta \in [0,1]\}$ since for $\beta \in [0,1]$ the term

$$P \cdot (-i\beta^{-1}) = \frac{1+\beta}{1-\beta}$$

ranges over $[1, \infty]$. The statement follows in the same way as above. □

Now, it is not difficult to classify the one-dimensional subspaces of $\mathbb{C}^3$ up to rotations from SO(3).

Lemma 9.3.4. *Let $S \subset \mathbb{C}^3$ be a one-dimensional subspace. Then there is a unique $\beta \in [0,1]$ such that*

$$S = R \,\mathrm{span} \left\{ \begin{pmatrix} 1 \\ i\beta \\ 0 \end{pmatrix} \right\}$$

holds for some rotation $R \in \mathrm{SO}(3)$.

Proof. Let $S = \mathrm{span}\{v\}$ for some non-zero vector $v \in \mathbb{C}^3$, and write $v = x + iy$ for real vectors $x, y \in \mathbb{R}^3$. Clearly, there is a rotation $R_1 \in \mathrm{SO}(3)$ such that $R_1 x \in \mathrm{span}\{e_1\}$.

Let $R_1 y =: \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix}^t$. Then there is a rotation $R_2 \in \mathrm{SO}(2)$ which sends $\begin{pmatrix} y_2 & y_3 \end{pmatrix}^t$ into $\operatorname{span}\left\{ \begin{pmatrix} 1 & 0 \end{pmatrix}^t \right\}$. Consequently, we have

$$\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & R_2 \end{array}\right] R_1 v = \begin{pmatrix} a+ib \\ ic \\ 0 \end{pmatrix}$$

for some real numbers $a, b, c \in \mathbb{R}$. Finally, by the previous lemma there exists some orthogonal matrix $R_3 \in \mathrm{O}(2)$ and unique $\beta \in [0,1]$ such that

$$R_3 \begin{pmatrix} a+ib \\ ic \end{pmatrix} \in \operatorname{span}\left\{ \begin{pmatrix} 1 \\ i\beta \end{pmatrix} \right\}.$$

By choosing $\kappa \in \{1, -1\}$ and $\lambda \in \mathbb{C}$ suitably we obtain

$$\left[\begin{array}{c|c} R_3 & 0 \\ \hline 0 & \kappa \end{array}\right] \in \mathrm{SO}(3) \quad \text{and} \quad \lambda \left[\begin{array}{c|c} R_3 & 0 \\ \hline 0 & \kappa \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & R_2 \end{array}\right] R_1 v = \begin{pmatrix} 1 \\ i\beta \\ 0 \end{pmatrix}.$$

The statement follows immediately. □

Theorem 9.3.5. *Every directed loopfree quantum graph on $(M_2, 2\mathrm{Tr})$ with one-dimensional quantum edge space is isomorphic to exactly one of the quantum graphs $\mathcal{G}_\beta$ with quantum edge space V_β given by*

$$V_\beta := \operatorname{span}\{\sigma_1 + i\beta\sigma_2\}$$

for some $\beta \in [0,1]$. Thus, every loopfree directed quantum graph on $(M_2, 2\mathrm{Tr})$ is isomorphic to exactly one of the quantum graphs with the following quantum adjacency *matrices:*

$$\begin{aligned}
A_0 &= 0,\\
A_1^{(\beta)} &= \frac{1}{1+\beta^2}(\sigma_1 + i\beta\sigma_2) \otimes \overline{(\sigma_1 + i\beta\sigma_2)},\\
&= \frac{1}{1+\beta^2}\begin{pmatrix} 0 & 0 & 0 & \beta_+^2 \\ 0 & 0 & \beta_+\beta_- & 0 \\ 0 & \beta_+\beta_- & 0 & 0 \\ \beta_-^2 & 0 & 0 & 0 \end{pmatrix},\\
A_2^{(\beta)} &= \sigma_3 \otimes \overline{\sigma_3} + \frac{1}{1+\beta^2}(i\beta\sigma_1 + \sigma_2) \otimes \overline{(i\beta\sigma_1 + \sigma_2)},\\
&= \frac{1}{1+\beta^2}\begin{pmatrix} 0 & 0 & 0 & \beta_-^2 \\ 0 & 0 & -\beta_+\beta_- & 0 \\ 0 & -\beta_+\beta_- & 0 & 0 \\ \beta_+^2 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\\
A_3 &= \sigma_1 \otimes \overline{\sigma_1} + \sigma_2 \otimes \overline{\sigma_2} + \sigma_3 \otimes \overline{\sigma_3}\\
&= \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix},
\end{aligned}$$

where $\beta_+ := 1 + \beta$ and $\beta_- := 1 - \beta$ for $\beta \in [0, 1]$. As before, we express a map from M_2 into M_2 as a 4×4-matrix with respect to the basis $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ of M_2 in line with Remark 9.1.2.

Proof. Assume we are given a loopfree quantum graph on $(M_2, 2\mathrm{Tr})$ with edge space $V \subset M_2$. If $V = \{0\}$ or $V = M_2$, then the quantum graph is one of the quantum graphs with quantum adjacency matrix A_0 or A_3 from Theorem 9.3.1. Only two cases remain: V is one-dimensional or two-dimensional. In the first case, the previous lemma yields a unique $\beta \in [0, 1]$ such that up to an isomorphism, i.e. conjugation with a unitary, we have $V = \mathrm{span}\{\sigma_1 + i\beta\sigma_2\}$. The matrix

$$\sigma_1 + i\beta\sigma_2 = \begin{pmatrix} 0 & 1+\beta \\ 1-\beta & 0 \end{pmatrix}$$

has norm $\sqrt{1+\beta^2}$ with respect to the GNS inner product induced by tr; hence, by Proposition 9.1.1 the quantum adjacency matrix of the associated quantum graph is given by

$$\frac{1}{1+\beta^2}(\sigma_1 + i\beta\sigma_2) \otimes \overline{(\sigma_1 + i\beta\sigma_2)}.$$

Finally, if V is two-dimensional, then V is the orthogonal complement of a one-dimensional subspace $V \subset M_2$ in $\mathrm{span}(I_2)^\perp \subset M_2$. In other words, up to an isomorphism, its Pauli edge space $S \subset \mathbb{C}^3$ is the orthogonal complement of

$$\operatorname{span}\left\{\begin{pmatrix} 1 \\ i\beta \\ 0 \end{pmatrix}\right\}$$

for a unique $\beta \in [0, 1]$. Hence, we can assume that S is equal to

$$S = \operatorname{span}\left\{\begin{pmatrix} i\beta \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

Thus, $V = \operatorname{span}\{i\beta\sigma_1 + \sigma_2, \sigma_3\}$, and a simple calculation yields the quantum adjacency matrix $A_2^{(\beta)}$ from the statement. $\square$

10. Classical derived graphs

The notion of a voltage graph and its derived graph was introduced in the 1980s by Gross and Tucker [43, 42] as a tool to construct covering graphs. Generally, voltage graphs are considered in the context of directed multigraphs. A directed multigraph Γ is given by a (finite) vertex set V and a (finite) edge set E together with maps $s, t : E \to V$ which assign to every edge its starting vertex and terminal vertex, respectively. If $s(e) = t(e)$, then the edge e is a loop. If $s(e) = v \in V$ and $t(e) = u \in V$, we suggestively write $e : v \to u$. Let us recall the following definitions from Gross and Tucker.

Definition 10.0.1 ([42, Section 2.1.1])**.** A *voltage graph* is a pair (Γ, λ) consisting of a (not necessarily simple) finite directed graph Γ with vertex set V and edge set E, together with a labeling $\lambda : E \to G$ of the edges of Γ by elements of a finite group G. We also call (Γ, λ) a G-voltage graph.

The *(right) derived graph* Γ^λ of (Γ, λ) has vertex set $V \times G$ and edge set $E \times G$. Whenever $e \in E$ is an edge from u to v in Γ and $g \in G$ is a group element, then the edge (e, g) goes from (u, g) to $(v, g\lambda(e))$ in Γ^λ, i.e.

$$e : u \to v \text{ in } \Gamma \text{ and } g \in G \quad \Longrightarrow \quad (e, g) : (u, g) \to (v, g\lambda(e)) \text{ in } \Gamma^\lambda.$$

Whenever Γ^λ is the derived graph of a voltage graph (Γ, λ), then there is a natural surjective graph homomorphism

$$f : \Gamma^\lambda \to \Gamma, (v, g) \mapsto v, (e, g) \mapsto e, \qquad \text{for all } v \in V, g \in G.$$

Thus, derived graphs are special cases of covering graphs.

Example 10.0.2. As an example, a directed version of the Petersen graph can be constructed as the derived graph of the "dumbbell" graph with voltages from the cyclic groups $\mathbb{Z}_5$ as depicted in Figure 10.1.

Later we will present a construction for quantum graphs that generalizes the classical derived graph construction. However, we will only consider the situation where the derived graph is simple. The next proposition characterizes when this is the case.

Proposition 10.0.3. *The derived graph Γ^λ of a voltage graph (Γ, λ) is simple if and only if for any two vertices $u, v \in V$ and group element $g \in G$ there is at most one edge $e : u \to v$ that is labeled by g.*

Proof. Assume that Γ^λ is not simple, and let $(e, g), (f, h)$ be two edges that have the same starting vertex and the same terminal vertex in Γ^λ. It is not difficult to check that this implies that $g = h$ and that e and f have the same starting vertex and the same terminal vertex in Γ as well. Further, they must be labeled by the same group element. Thus, the statement follows by contraposition. ☐

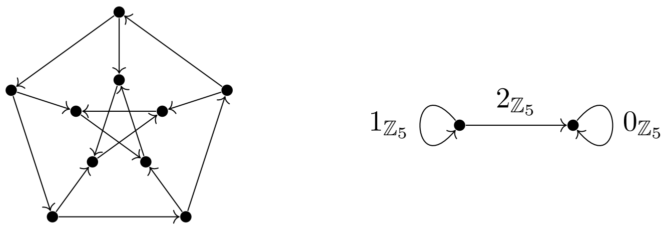


Figure 10.1.: The Petersen graph is the derived graph of a voltage graph over the group $\mathbb{Z}_5$.

If Γ^λ is the derived graph of a G-voltage graph (Γ, λ), then it is not difficult to see that there is a free action of the group G on Γ^λ. Indeed, if V (E) is the vertex (edge) set of Γ, then the vertex (edge) set of Γ^λ is $V \times G$ ($E \times G$) and we can define an action $\alpha : G \to \mathrm{Aut}(\Gamma^\lambda)$ by

$$\alpha_h(v, g) = (v, hg), \quad \alpha_h(e, g) = (e, hg),$$

for all $v \in V, e \in E, g, h \in G$. It is not hard to check that α_h is a graph automorphism of Γ^λ. Further, this action α is free which means that α_h has no fixed-points unless h is the identity in G. The *Gross–Tucker theorem* states that a graph Γ is the derived graph of a G-voltage graph precisely if it has a free action of G.

Theorem 10.0.4 ([42, Theorem 2.2.2])**.** *Let Γ be a finite graph, and assume that there is a free action $\alpha : G \to \mathrm{Aut}(\Gamma)$ of a (finite) group G on Γ. Then there exists a voltage assignment $\lambda : E_{\Gamma/G} \to G$ on the quotient graph Γ/G such that*

$$\Gamma \cong (\Gamma/G)^\lambda.$$

Proof. Let us briefly sketch the construction of the voltage graph $(\Gamma/G, \lambda)$. The action α of G partitions the vertices of Γ into orbits which are all of the same size as G because the action is free. Thus, one can label the vertices of Γ by pairs (v, g) consisting of a vertex of the quotient graph v and a group element g such that $\alpha_h(v, g) = (v, hg)$ holds for all $v \in V_{\Gamma/G}$ and $h, g \in G$. In the same way, the edges are partitioned into orbits of the same size as G. If $e : v \mapsto u$ is an edge in the quotient graph then its pre-image under the quotient map consists of $|G|$ edges, each one starting at a different vertex of the form (v, g) for $g \in G$. For all $g \in G$, we label the edge starting from $g \in G$ by (e, g). It only remains to determine the voltage assignment on the edges of the quotient graph. If $e : u \mapsto v$ is an edge in the quotient graph, then the edge $(e, 1)$ in the preimage goes from $(v, 1)$ to some vertex (u, g). Then, one assigns the voltage $\lambda(e) := g$. $\square$

11. Crossed product quantum sets

Assume that X is a finite set together with a free action $\alpha : G \to \mathrm{Aut}(X)$ of a finite group G on X. Evidently, we have an isomorphism of finite sets

$$X \cong (X/G) \times G, \tag{11.1}$$

since every orbit of the group action must have the same size as G. Further, the isomorphism can be chosen such that it is G-equivariant with respect to the natural action of G on $X/G \times G$ by left multiplication, i.e. such that it satisfies

$$x \mapsto (y, g) \implies \alpha_h(x) \mapsto (y, hg)$$

for all $x \in X, y \in X/G$ and $h, g \in G$. The classical Gross–Tucker theorem builds on this isomorphism.

For quantum sets we will consider a generalization of (11.1). The starting point of this is a quantum set (B, ψ) together with an action $\alpha : G \to \mathrm{Aut}(B, \psi)$ of a finite abelian group G.[1] Then, we describe a crossed product construction for quantum sets such that there is a (quantum set) isomorphism

$$(B, \psi) \cong (\tilde{B}, \tilde{\psi}) \rtimes_{\hat{\alpha}} \hat{G}, \tag{11.2}$$

that is G-equivariant with respect to the given action α on the left-hand side and a natural action of G on the right-hand side. Crucially, as in the classical case such an isomorphism exists only for suitable actions α. Whereas in the classical situation the action needs to be free, in the quantum case our conditions on α are derived from a theorem of Landstad that characterizes crossed product C*-algebras.

In what follows we first introduce some notations for crossed products of C*-algebras and then discuss how we can turn a crossed product C*-algebra into a quantum set in a natural way. Remarkably we will see that crossed product quantum sets with respect to different actions of a fixed group are generally not isomorphic, but they are always *quantum* isomorphic. In particular, we have

$$(\tilde{B}, \tilde{\psi}) \rtimes_{\hat{\alpha}} \hat{G} \cong_q (\tilde{B}, \tilde{\psi}) \otimes (C(G), \psi_G)$$

for all actions $\hat{\alpha}$ and a natural definition of the tensor product quantum set (see Example 11.2.5). Finally, we use a theorem of Landstad to prove (11.2) for suitable actions α on (B, ψ).

[1] We restrict to finite abelian groups because our constructions involve the dual group $\hat{G}$. In an ongoing work which is not part of this thesis, we generalize the results to arbitrary (finite) groups G by working in the class of finite compact quantum groups which is closed under the taking duals.

11.1. Crossed products of C*-algebras

We refer the reader to Blackadar's book [10] for a general reference on crossed product C*-algebras. If $\tilde{B}$ is a finite-dimensional C*-algebra and $\hat{\alpha} : \hat{G} \to \mathrm{Aut}(\tilde{B})$ is a group action of a finite group on B, then the crossed product $B \rtimes_{\hat{\alpha}} \hat{G}$ is again a finite-dimensional C*-algebra. This is the situation which we will encounter throughout the rest of this thesis. We will consistently use the following notation.

Notation 11.1.1. Let $(\tilde{B}, \tilde{\psi})$ be a quantum set and let $\hat{\alpha} : \hat{G} \to \mathrm{Aut}(\tilde{B}, \tilde{\psi})$ be an action of $\hat{G}$, the dual of a finite abelian group G. Further, let

$$B := \tilde{B} \rtimes_{\hat{\alpha}} \hat{G}$$

be the associated crossed product C*-algebra. There is a representation

$$(u_\chi)_{\chi \in \hat{G}} \subset B \quad \text{with} \quad \hat{\alpha}_\chi(b) = u_\chi b u_\chi^* \text{ for all } b \in \tilde{B}, \chi \in \hat{G},$$

such that every element of B can be written uniquely as

$$b = \sum_{\chi \in \hat{G}} b_\chi u_\chi \qquad \text{for suitable } b_\chi \in \tilde{B}.$$

Further, multiplication and involution on B are determined by

$$\begin{aligned} (b_\chi u_\chi)(c_\xi u_\xi) &= b_\chi \hat{\alpha}_\chi(c_\xi) u_{\chi\xi}, \\ (b_\chi u_\chi)^* &= \hat{\alpha}_{\chi^{-1}}(b_\chi^*) u_{\chi^{-1}}. \end{aligned}$$

Finally, there is a linear map $E_\rtimes : B \to \tilde{B}$ given by

$$E_\rtimes \left(\sum_{\chi \in \hat{G}} b_\chi u_\chi \right) = n b_1,$$

where $n := |G| = |\hat{G}|$. This is a scaled version of the natural conditional expectation from B onto $\tilde{B}$. The scaling is chosen so that it fits with the quantum set functionals that we will associate with B and $\tilde{B}$.

11.2. A quantum set functional on the crossed product C*-algebra

Assume we have given a finite-dimensional C*-algebra $\tilde{B}$ together with a group action $\hat{\alpha} : \hat{G} \to \mathrm{Aut}(\tilde{B})$ of $\hat{G}$, the dual of a finite abelian group G. The crossed product construction is an operation

$$\tilde{B} \mapsto B := \tilde{B} \rtimes_{\hat{\alpha}} \hat{G},$$

which produces a new C*-algebra $B = \tilde{B} \rtimes_{\hat{\alpha}} \hat{G}$ from the pair $(\tilde{B}, \hat{\alpha})$.

What happens if $\tilde{B}$ is equipped with a functional $\tilde{\psi}$ such that $(\tilde{B}, \tilde{\psi})$ is a quantum set? In this section, we discuss how one can construct a functional $\psi_{\rtimes}$ on the crossed product C*-algebra $\tilde{B} \rtimes_{\hat{\alpha}} \hat{G}$ such that $(\tilde{B} \rtimes_{\hat{\alpha}} \hat{G}, \psi_{\rtimes})$ is again a quantum set. This gives us an operation

$$(\tilde{B}, \tilde{\psi}) \mapsto (\tilde{B}, \tilde{\psi}) \rtimes_{\hat{\alpha}} \hat{G} =: (B, \psi_{\rtimes}),$$

which produces a new quantum set $(B, \psi_{\rtimes})$, which we call the *crossed product quantum set*, from a quantum set $(\tilde{B}, \tilde{\psi})$ together with a group action $\hat{\alpha}$.

Definition 11.2.1. Let $(\tilde{B}, \tilde{\psi})$ be a quantum set and let $\hat{\alpha} : \hat{G} \to \mathrm{Aut}(\tilde{B}, \tilde{\psi})$ be an action of $\hat{G}$, the dual of a finite abelian group G. The *crossed product quantum set* is defined as

$$(\tilde{B}, \tilde{\psi}) \rtimes_{\hat{\alpha}} \hat{G} := \left(\tilde{B} \rtimes_{\hat{\alpha}} \hat{G}, \psi_{\rtimes}\right),$$

where $\tilde{B} \rtimes_{\hat{\alpha}} \hat{G}$ is the crossed product C*-algebra and $\psi_{\rtimes}$ is the functional given by

$$\psi_{\rtimes}\left(\sum_{\chi \in \hat{G}} b_{\chi} u_{\chi}\right) = n\tilde{\psi}(b_1)$$

with $n := |G| = |\hat{G}|$. A natural action of G on $(B, \psi) \rtimes_{\hat{\alpha}} \hat{G}$ is given by

$$\alpha_g\left(\sum_{\chi \in \hat{G}} b_{\chi} u_{\chi}\right) = \sum_{\chi \in \hat{G}} \chi(g) b_{\chi} u_{\chi}.$$

We will proceed to check that the above definition makes sense, i.e. we verify that $\psi_{\rtimes}$ is indeed a quantum set functional on the C*-algebra $\tilde{B} \rtimes_{\hat{\alpha}} \hat{G}$. Before we prove a useful lemma, let us introduce some notation.

Notation 11.2.2. Given a quantum set $(\tilde{B}, \tilde{\psi})$ and an action $\hat{\alpha} : \hat{G} \to \mathrm{Aut}(\tilde{B}, \tilde{\psi})$ as above, we denote

- by $\tilde{m}$ the multiplication map on $\tilde{B}$ and by $\tilde{m}^*$ its adjoint with respect to the inner product induced by $\tilde{\psi}$,
- by $m_{\rtimes}$ the multiplication map on $\tilde{B} \rtimes_{\hat{\alpha}} \hat{G}$ and by $m_{\rtimes}^*$ its adjoint with respect to the inner product induced by $\psi_{\rtimes}$,
- by $E_{\rtimes}$ the linear map
$$E_{\rtimes} : \tilde{B} \rtimes_{\hat{\alpha}} \hat{G} \to \tilde{B}, \quad \sum_{\chi \in \hat{G}} b_{\chi} u_{\chi} \mapsto n b_1,$$
and by $E_{\rtimes}^*$ its adjoint with respect to the inner products induced by the functionals $\tilde{\psi}$ and $\psi_{\rtimes}$ as above.

Lemma 11.2.3. *Let $\hat{\alpha}$ be an action of $\hat{G}$ on the quantum set $(\tilde{B}, \tilde{\psi})$ as above. Then the following formulas are true for all $b, c \in \tilde{B}$ and $\chi, \xi \in \hat{G}$:*

$$\langle bu_\chi, cu_\xi \rangle_{\psi_\rtimes} = n\delta_{\chi=\xi} \langle b, c \rangle_{\tilde{\psi}}, \tag{11.3}$$

$$m^*_\rtimes (bu_\chi) = \frac{1}{n} \sum_{\xi \in \hat{G}} \sum_i b_i^{(1)} u_\xi \otimes \hat{\alpha}_{\xi^{-1}}(b_i^{(2)}) u_{\xi^{-1}\chi}, \tag{11.4}$$

$$E^*_\rtimes(b) = b, \tag{11.5}$$

where in (11.4) *the elements $b_i^{(1)}, b_i^{(2)} \in \tilde{B}$ are such that $\tilde{m}^*(b) = \sum_i b_i^{(1)} \otimes b_i^{(2)}$.*

Proof. The first statement follows from

$$\langle bu_\chi, cu_\xi \rangle_{\psi_\rtimes} = \psi_\rtimes(u_\chi^* b^* cu_\xi) = n\delta_{\chi=\xi} \tilde{\psi}(\hat{\alpha}_{\chi^{-1}}(b^*c)) = n\delta_{\chi=\xi} \langle b, c \rangle_{\tilde{\psi}},$$

where we use $\tilde{\psi} = \tilde{\psi} \circ \hat{\alpha}.$ in the last step. Using (11.3), we get for any elements $bu_\chi, cu_\rho, du_\sigma \in \tilde{B} \rtimes_{\hat{\alpha}} \hat{G}$,

$$\begin{aligned} &\frac{1}{n} \sum_\xi \sum_i \langle b_i^{(1)} u_\xi \otimes \hat{\alpha}_{\xi^{-1}}(b_i^{(2)}) u_{\xi^{-1}\chi}, cu_\rho \otimes du_\sigma \rangle_{\psi_\rtimes \otimes \psi_\rtimes} \\ &\qquad = \frac{1}{n} \sum_\xi \sum_i \langle b_i^{(1)} u_\xi, cu_\rho \rangle_{\psi_\rtimes} \langle \hat{\alpha}_{\xi^{-1}}(b_i^{(2)}) u_{\xi^{-1}\chi}, du_\sigma \rangle_{\psi_\rtimes} \\ &\qquad = n \sum_\xi \sum_i \delta_{\xi=\rho} \langle b_i^{(1)}, c \rangle_{\tilde{\psi}} \delta_{\xi^{-1}\chi=\sigma} \langle \hat{\alpha}_{\xi^{-1}}(b_i^{(2)}), d \rangle_{\tilde{\psi}}. \end{aligned}$$

As $\tilde{\psi} = \tilde{\psi} \circ \hat{\alpha}.$ we have

$$\langle \hat{\alpha}_{\xi^{-1}}(b_i^{(2)}), d \rangle_{\tilde{\psi}} = \langle b_i^{(2)}, \hat{\alpha}_\xi(d) \rangle_{\tilde{\psi}}.$$

Hence, we can continue the computation from above with

$$\begin{aligned} \ldots &= n\delta_{\chi=\rho\sigma} \sum_i \langle b_i^{(1)}, c \rangle_{\tilde{\psi}} \langle b_i^{(2)}, \hat{\alpha}_\rho(d) \rangle_{\tilde{\psi}} \\ &= n\delta_{\chi=\rho\sigma} \langle m^*_\rtimes(b), c \otimes \hat{\alpha}_\rho(d) \rangle_{\tilde{\psi} \otimes \tilde{\psi}} \\ &= n\delta_{\chi=\rho\sigma} \langle b, \tilde{m}(c \otimes \hat{\alpha}_\rho(d)) \rangle_{\tilde{\psi}} \\ &= \langle bu_\chi, m_\rtimes(cu_\rho \otimes du_\sigma) \rangle_{\psi_\rtimes}. \end{aligned}$$

Altogether this shows $m^*_\rtimes(bu_\chi) = \frac{1}{n} \sum_{\xi \in \hat{G}} \sum_i b_i^{(1)} u_\xi \otimes \hat{\alpha}_{\xi^{-1}}(b_i^{(2)}) u_{\xi^{-1}\chi}$.

It remains to prove formula (11.5) for the adjoint $E_\rtimes$. For this, let $b, c \in \tilde{B}$ and $\chi \in \hat{G}$ and use (11.3) to observe

$$\langle E_\rtimes (bu_\chi), c \rangle_{\tilde{\psi}} = n\delta_{\chi=1} \langle b, c \rangle_{\tilde{\psi}} = \langle bu_\chi, c \rangle_{\psi_\rtimes} = \langle bu_\chi, E^*_\rtimes(c) \rangle_{\psi_\rtimes}.$$

□

Proposition 11.2.4. *The map $\psi_\rtimes$ is a quantum set functional on the C*-algebra $\tilde{B}\rtimes_{\hat{\alpha}}\hat{G}$ if and only if $\tilde{\psi}$ is a quantum set functional on $\tilde{B}$.*

Proof. Using (11.4) from Lemma 11.2.3 observe that for $b \in \tilde{B}$ and $\chi \in \hat{G}$,

$$\begin{aligned} m_\rtimes m_\rtimes^* (bu_\chi) &= \frac{1}{n}\sum_{\xi\in\hat{G}}\sum_i b_i^{(1)} u_\xi \hat{\alpha}_{\xi^{-1}}(b_i^{(2)}) u_{\xi^{-1}\chi} \\ &= \sum_i b_i^{(1)} b_i^{(2)} u_\chi \end{aligned}$$

is equal to bu_χ if and only if $\sum_i b_i^{(1)} b_i^{(2)} = b$ holds. Thus, we have that $m_\rtimes m_\rtimes^* = \mathrm{Id}_{\tilde{B}\rtimes_{\hat{\alpha}}\hat{G}}$ is equivalent to $\tilde{m}\tilde{m}^* = \mathrm{Id}_{\tilde{B}}$. Evidently, $\psi_\rtimes$ is positive and faithful if and only if the same holds for $\tilde{\psi}$. Thus, the former is a quantum set functional if and only if the latter one is. □

If $\hat{\alpha}$ is the trivial action, then it is well known that the crossed product is simply a tensor product. Let us describe this situation in more detail to prepare for later calculations.

Example 11.2.5. Assume that $\mathrm{triv} : \hat{G} \to \mathrm{Aut}(\tilde{B}, \tilde{\psi})$ is the trivial action given by

$$\mathrm{triv}_\chi(b) = b \quad \text{for all } b \in \tilde{B}, \chi \in \hat{G}.$$

Then the crossed product $\tilde{B} \rtimes_{\mathrm{triv}} \hat{G}$ is actually isomorphic to the tensor product C*-algebra $\tilde{B} \otimes C(G)$, and an isomorphism is given by

$$\begin{aligned} \tilde{B} \rtimes_{\mathrm{triv}} \hat{G} &\to \tilde{B} \otimes C(G), \\ \sum_{\chi\in\hat{G}} b_\chi u_\chi &\mapsto \sum_{\chi\in\hat{G}} b_\chi \otimes \tilde{u}_\chi, \end{aligned}$$

where $\tilde{u}_\chi = (\chi(g))_{g\in G} \in C(G)$ is the function on G which evaluates to $\chi(g)$ at $g \in G$. Recall the functional ψ_G on $C(G)$ from Example 8.1.4 which is given by, equivalently,

$$\psi_G(e_g) = 1 \text{ for all } g \in G \quad \text{or} \quad \psi_G(\tilde{u}_\chi) = n\delta_{\chi=1} \text{ for all } \chi \in \hat{G},$$

where $n = |G|$. The C*-algebra isomorphism from above identifies the functionals $\psi_\rtimes$ and $\tilde{\psi} \otimes \psi_G$. Indeed, we have for every $b \in \tilde{B}$ and $\chi \in \hat{G}$,

$$\psi_\rtimes(bu_\chi) = \delta_{\chi=1} n\tilde{\psi}(b),$$

as well as

$$(\tilde{\psi} \otimes \psi_G)(b \otimes \tilde{u}_\chi) = \delta_{\chi=1} n\tilde{\psi}(b).$$

Summarizing, for the trivial action we have an isomorphism of quantum sets

$$(\tilde{B} \otimes C(G), \tilde{\psi} \otimes \psi_G) \cong (\tilde{B} \rtimes_{\mathrm{triv}} \hat{G}, \psi_\rtimes).$$

We denote the quantum set on the left-hand side by $(\tilde{B}, \tilde{\psi}) \otimes (C(G), \psi_G)$ and call it the *tensor product quantum set* of $(\tilde{B}, \tilde{\psi})$ and $(C(G), \psi_G)$. Later we will prove the remarkable fact that for *any* action $\hat{\alpha}$, which is not necessarily trivial, one has a *quantum* isomorphism $(\tilde{B}, \tilde{\psi}) \otimes (C(G), \psi_G) \cong (\tilde{B}, \tilde{\psi}) \rtimes_{\hat{\alpha}} \hat{G}$.

11.3. Quantum isomorphisms between crossed product quantum sets

Let us start with some notation. For a (finite abelian) group G the C*-algebra $C(G)$ has two canonical bases: First, the basis $(e_g)_{g\in G}$ given by the indicator functions, and second the basis $(\tilde{u}_\chi)_{\chi\in\hat{G}}$ given by the functions with $\tilde{u}_\chi(g) = \chi(g)$ for all $g \in G$. The algebra $C(G)$ becomes a quantum set when it is equipped with the functional ψ_G given by, equivalently,

$$\psi_G(e_g) = 1 \text{ for all } g \in G, \qquad \text{or} \qquad \psi_G(\tilde{u}_\chi) = n\delta_{\chi=1} \text{ for all } \chi \in \hat{G},$$

where $n = |G| = |\hat{G}|$ and $1 \in \hat{G}$ is the trivial character, see also Example 8.1.4. Further, the functional ψ_G induces an inner product on $C(G)$ which makes the latter into a Hilbert space denoted by $L^2(G)$. The elements of $B(L^2(G))$ can be identified with $\hat{G}$-index matrices, and a basis for these is given by the standard matrix units $E_{\xi,\zeta}$ for $\xi, \zeta \in \hat{G}$. To make the notation better readable we write e_χ for $\tilde{u}_\chi \in C(G)$ when we consider it as an element of the Hilbert space $L^2(G)$. Yet we remind the reader that $\langle e_\chi, e_\xi\rangle = \psi_G(\tilde{u}_\chi^*\tilde{u}_\xi) = n\delta_{\chi=\xi}$ for all $\chi, \xi \in \hat{G}$.

Proposition 11.3.1. *Let $\hat{\alpha} : \hat{G} \to \operatorname{Aut}(\tilde{B}, \tilde{\psi})$ be an action of $\hat{G}$, the dual of a finite abelian group G, on a quantum set $(\tilde{B}, \tilde{\psi})$, and let $\mathcal{H} := L^2(G)$. Then, the map*

$$\rho : \begin{cases} \tilde{B} \otimes C(G) \to \left(\tilde{B} \rtimes_{\hat{\alpha}} \hat{G}\right) \otimes B(\mathcal{H}), \\ \qquad b \otimes \tilde{u}_\chi \mapsto \displaystyle\sum_{\zeta\in\hat{G}} u_\zeta b u_\zeta^* u_\chi \otimes E_{\zeta,\chi^{-1}\zeta} \end{cases}$$

is a quantum isomorphism

$$(\tilde{B}, \tilde{\psi}) \rtimes_{\hat{\alpha}} \hat{G} \cong_q (\tilde{B}, \tilde{\psi}) \otimes (C(G), \psi_G).$$

Proof. To prove that ρ is a $*$-homomorphism, we construct it via the universal property of the tensor product. Let $(\hat{u}_\chi)_{\chi\in\hat{G}}$ be the natural representation of $\hat{G}$ on $\mathcal{H}$ given by

$$\hat{u}_\chi e_\xi = e_{\chi\xi} \qquad \text{for all } \chi, \xi \in \hat{G},$$

and consider the maps

$$\begin{aligned} \rho_1 : \tilde{B} &\to \left(\tilde{B} \rtimes_{\hat{\alpha}} \hat{G}\right) \otimes B(\mathcal{H}), & b &\mapsto \sum_{\chi\in\hat{G}} u_\chi b u_\chi^* \otimes E_{\chi,\chi}, \\ \rho_2 : C(G) &\to \left(\tilde{B} \rtimes_{\hat{\alpha}} \hat{G}\right) \otimes B(\mathcal{H}), & \tilde{u}_\chi &\mapsto u_\chi \otimes \hat{u}_\chi. \end{aligned}$$

It is not hard to see that ρ_1 and ρ_2 are unital $*$-homomorphisms. Furthermore, they have commuting ranges since for all $b \in \tilde{B}$ and $\chi \in \hat{G}$,

$$\begin{aligned} \rho_2(\tilde{u}_\chi)\rho_1(b)\rho_2(\tilde{u}_\chi)^* &= \sum_{\xi\in\hat{G}} u_\chi u_\xi b u_\xi^* u_\chi^* \otimes \hat{u}_\chi E_{\xi,\xi} \hat{u}_\chi^* \\ &= \sum_{\zeta\in\hat{G}} u_\zeta b u_\zeta^* \otimes E_{\zeta,\zeta} = \rho_1(b). \end{aligned}$$

Therefore, the universal property of the tensor product of C*-algebras yields a unique unital $*$-homomorphism

$$\rho : \tilde{B} \otimes C(G) \to \left(\tilde{B} \rtimes_{\hat{\alpha}} \hat{G}\right) \otimes B(\mathcal{H}),$$

such that $\rho|_{\tilde{B}} = \rho_1$ and $\rho|_{C(G)} = \rho_2$. A short calculation confirms that this $*$-homomorphism is the map ρ from the statement.

It remains to show that the associated map

$$p : L^2(\tilde{B} \otimes C(G)) \otimes \mathcal{H} \to L^2(\tilde{B} \rtimes_{\hat{\alpha}} \hat{G}) \otimes \mathcal{H}$$

from Definition 8.3.1 is unitary. First, let us show that the adjoint of p is given by

$$p^* : \begin{cases} L^2(\tilde{B} \rtimes_{\hat{\alpha}} \hat{G}) \otimes \mathcal{H} \to L^2(\tilde{B} \otimes C(G)) \otimes \mathcal{H}, \\ \qquad\qquad b_\chi u_\chi \otimes e_\xi \mapsto u_\xi^* b_\chi u_\xi \otimes \tilde{u}_\chi \otimes e_{\chi^{-1}\xi}. \end{cases} \tag{11.6}$$

Indeed, for all $c \in \tilde{B}$ and $\alpha, \beta \in \hat{G}$ we have

$$p(c \otimes \tilde{u}_\alpha \otimes e_\beta) = \sum_{\gamma \in \hat{G}} u_\gamma c u_\gamma^* u_\alpha \otimes E_{\gamma, \alpha^{-1}\gamma} e_\beta = u_{\alpha\beta} c u_\beta^* \otimes e_{\alpha\beta}.$$

It follows for all $b_\chi \in \tilde{B}$ and $\chi, \xi \in \hat{G}$,

$$\begin{aligned} &\langle b_\chi u_\chi \otimes e_\xi, p(c \otimes \tilde{u}_\alpha \otimes e_\beta)\rangle_{\psi_\rtimes \otimes \psi_G} \\ &\quad = \langle b_\chi u_\chi \otimes e_\xi, u_{\alpha\beta} c u_\beta^* \otimes e_{\alpha\beta}\rangle_{\psi_\rtimes \otimes \psi_G} \\ &\quad = n\delta_{\xi = \alpha\beta} \langle b_\chi u_\chi, u_{\alpha\beta} c u_\beta^*\rangle_{\psi_\rtimes}. \end{aligned}$$

Using formula (11.3) for $\langle \cdot, \cdot \rangle_{\psi_\rtimes}$, the previous expression becomes

$$\langle b_\chi u_\chi \otimes e_\xi, p(c \otimes \tilde{u}_\alpha \otimes e_\beta)\rangle_{\psi_\rtimes \otimes \psi_G} = n^2 \delta_{\xi = \alpha\beta} \delta_{\chi = \alpha} \langle b_\chi, \hat{\alpha}_{\alpha\beta}(c)\rangle_{\tilde{\psi}}. \tag{11.7}$$

On the other hand, for the map p^* from (11.6) we compute

$$\begin{aligned} &\langle p^*(b_\chi u_\chi \otimes e_\xi), c \otimes \tilde{u}_\alpha \otimes e_\beta\rangle_{\tilde{\psi} \otimes \psi_G \otimes \psi_G} \\ &\qquad = \langle u_\xi^* b_\chi u_\xi \otimes \tilde{u}_\chi \otimes e_{\chi^{-1}\xi}, c \otimes \tilde{u}_\alpha \otimes e_\beta\rangle_{\tilde{\psi} \otimes \psi_G \otimes \psi_G} \\ &\qquad = \langle u_\xi^* b_\chi u_\xi, c\rangle_{\tilde{\psi}} \langle \tilde{u}_\chi, \tilde{u}_\alpha\rangle_{\psi_G} \langle e_{\chi^{-1}\xi}, e_\beta\rangle_{\psi_G} \\ &\qquad = n^2 \delta_{\chi = \alpha} \delta_{\chi^{-1}\xi = \beta} \langle u_\xi^* b_\chi u_\xi, c\rangle_{\tilde{\psi}}. \end{aligned}$$

Using $\tilde{\psi} \circ \hat{\alpha}_\xi = \tilde{\psi}$ it is not hard to see

$$\langle u_\xi^* b_\chi u_\xi, c\rangle_{\tilde{\psi}} = \tilde{\psi}(u_\xi^* b_\chi^* u_\xi c) = \tilde{\psi}(b_\chi^* u_\xi c u_\xi^*) = \langle b_\chi, \hat{\alpha}_\xi(c)\rangle_{\tilde{\psi}}.$$

Since $\chi^{-1}\xi = \beta$ and $\chi = \alpha$ entail $\xi = \alpha\beta$ we arrive at

$$\langle p^*(b_\chi u_\chi \otimes e_\xi), c \otimes \tilde{u}_\alpha \otimes e_\beta\rangle_{\tilde{\psi} \otimes \psi_G \otimes \psi_G} = n^2 \delta_{\chi = \alpha} \delta_{\xi = \alpha\beta} \langle b_\chi, \hat{\alpha}_{\alpha\beta}(c)\rangle_{\tilde{\psi}}. \tag{11.8}$$

This is the same expression as in (11.7); consequently p^* is indeed the adjoint of the map p.

Finally, we are ready to show that p is a unitary map. It suffices to prove $pp^* = \mathrm{Id}$. For this, let $b_\chi u_\chi \in \tilde{B} \rtimes_{\hat{\alpha}} \hat{G}$ and $\xi \in \hat{G}$ and observe

$$\begin{aligned} pp^*(b_\chi u_\chi \otimes e_\xi) &= p\left(u_\xi^* b_\chi u_\xi \otimes \tilde{u}_\chi \otimes e_{\chi^{-1}\xi}\right) \\ &= \sum_{\gamma \in \hat{G}} u_\gamma u_\xi^* b_\chi u_\xi u_\gamma^* u_\chi \otimes E_{\gamma, \chi^{-1}\gamma} e_{\chi^{-1}\xi} \\ &= b_\chi u_\chi \otimes e_\xi. \end{aligned}$$

This finishes the proof. □

11.4. Landstad duality for quantum sets

Let us finally turn to the question of how the operation $(\tilde{B}, \tilde{\psi}) \mapsto (\tilde{B}, \tilde{\psi}) \rtimes_{\hat{\alpha}} \hat{G}$ can be inverted. Thus, given a quantum set (B, ψ) we want to find a smaller quantum set $(\tilde{B}, \tilde{\psi})$ together with an action $\hat{\alpha} : \hat{G} \to \mathrm{Aut}(\tilde{B}, \tilde{\psi})$ such that

$$(B, \psi) \cong (\tilde{B}, \tilde{\psi}) \rtimes_{\hat{\alpha}} \hat{G}.$$

This construction starts with an action $\alpha : G \to \mathrm{Aut}(B, \psi)$ of a group G on (B, ψ) where in our setting G is finite abelian and α satisfies a number of conditions. Then, the action $\hat{\alpha}$ is derived from α and the constructed isomorphism is G-equivariant with respect to the action α on the left-hand side and the natural action of G on the right-hand side.

For C*-algebras the problem is solved by Landstad's Duality Theorem [58]. This theorem works for general C*-algebras and von Neumann algebras, but we will only recall the particular version for finite-dimensional C*-algebras which is sufficient for our purposes.

Theorem 11.4.1 ([58, Theorem 2])**.** *Let B be a finite-dimensional C*-algebra and $\alpha : G \to \mathrm{Aut}(B)$ an action of a finite abelian group G on B. Then B is isomorphic to a crossed product C*-algebra over an action of $\hat{G}$ if and only if there exists a unitary representation $(u_\chi)_{\chi \in \hat{G}} \subset B$ such that*

$$\alpha_g(u_\chi) = \chi(g) u_\chi$$

holds for all $g \in G, \chi \in \hat{G}$. In this case, one has

$$B \cong B^\alpha \rtimes_{\hat{\alpha}} \hat{G},$$

where $B^\alpha = \{b \in B \mid \alpha_g(b) = b \text{ for all } g \in G\}$ is the fixed point algebra and $\hat{\alpha} : \hat{G} \to \mathrm{Aut}(B^\alpha)$ is the action defined by

$$\hat{\alpha}_\chi(b) = u_\chi b u_\chi^*.$$

The isomorphism is the identity on B^α and sends every u_χ to the unitary $\tilde{u}_\chi \in B^\alpha \rtimes_{\hat{\alpha}} \hat{G}$ that implements the automorphism $\hat{\alpha}_\chi$.

It is not difficult to obtain a version of Landstad Duality for quantum sets by combining the above theorem with our discussion of crossed product quantum sets.

Proposition 11.4.2. *Let (B, ψ) be a quantum set and $\alpha : G \to \mathrm{Aut}(B, \psi)$ be an action of a finite abelian group G on (B, ψ), which satisfies the conditions of Landstad's Duality Theorem (Theorem 11.4.1). Additionally, assume that*

$$\psi = \psi\,\mathrm{Ad}(u_\chi) \tag{11.9}$$

holds for all $\chi \in \hat{G}$, where $(u_\chi)_{\chi\in\hat{G}} \subset B$ is the unitary representation from Landstad's theorem. Then one has an isomorphism of quantum sets

$$(B, \psi) \cong (B^\alpha, \tilde{\psi}) \rtimes_{\hat{\alpha}} \hat{G},$$

where $\hat{\alpha} : \hat{G} \to \mathrm{Aut}(B^\alpha, \tilde{\psi})$ is the action from Landstad's theorem and $\tilde{\psi}$ is the quantum set functional on B^α given by

$$\tilde{\psi} := \frac{1}{n}\psi|_{B^\alpha}$$

with $n := |G| = |\hat{G}|$. Furthermore, this isomorphism is G-equivariant with respect to the action α of G on (B, ψ) and the natural action of G on $(B^\alpha, \tilde{\psi}) \rtimes_{\hat{\alpha}} \hat{G}$.

Proof. In view of Landstad's theorem, we only need to check that $\tilde{\psi}$ is a quantum set functional on B^α that is preserved by the action $\hat{\alpha}$, and that $\psi = \psi_\rtimes$ in the notation of the previous subsections.

First, let us check that $\tilde{\psi}$ is preserved by $\hat{\alpha}$ in the sense that

$$\tilde{\psi} = \tilde{\psi}\hat{\alpha}_\chi$$

holds for all $\chi \in \hat{G}$. Due to (11.9) we have for all $b \in B^\alpha$ and $\chi \in \hat{G}$

$$\tilde{\psi}(\hat{\alpha}_\chi(b)) = \frac{1}{n}\psi(u_\chi b u_\chi^*) = \frac{1}{n}\psi(b) = \tilde{\psi}(b).$$

Next, we verify $\psi = \psi_\rtimes$ where $\psi_\rtimes$ is derived from $\tilde{\psi}$ as in Definition 11.2.1. Thus, we need to check that

$$\psi(bu_\chi) = n\delta_{\chi=1}\tilde{\psi}(b)$$

holds for all $b \in B^\alpha$ and $\chi \in \hat{G}$. Let us first prove this equation. Since α is an action on the quantum set (B, ψ), we have for all $b \in B^\alpha$ and $\chi \in \hat{G}$

$$\psi(bu_\chi) = \psi(\alpha_g(bu_\chi)) = \chi(g)\psi(bu_\chi)$$

for all $g \in G$. Hence, one has $\psi(bu_\chi) = 0$ whenever $\chi \neq 1$. It follows directly that

$$\psi(bu_\chi) = \delta_{\chi=1}\psi(b) = n\delta_{\chi=1}\tilde{\psi}(b)$$

holds for all $b \in B^\alpha$ and $\chi \in \hat{G}$.

In fact, this already proves that $\tilde{\psi}$ is a quantum set functional on B^α: Proposition 11.2.4 guarantees that ψ is a quantum set functional on B if and only if $\tilde{\psi}$ is a quantum set functional on B^α.

Finally, let us check the G-equivariance of the isomorphism. In fact, every element of B can be written as a linear combination $\sum_{\chi\in\hat{G}} b_\chi u_\chi$ with $b_\chi \in B^\alpha$ and it suffices to note that the action α is given by

$$\alpha_g\left(\sum_{\chi\in\hat{G}} b_\chi u_\chi\right) = \sum_{\chi\in\hat{G}} b_\chi \alpha_g(u_\chi) = \sum_{\chi\in\hat{G}} \chi(g) b_\chi u_\chi.$$

□

12. Voltage and derived quantum graphs

In analogy with the classical theory of Gross and Tucker we introduce the notion of a *voltage quantum graph* and its associated *derived quantum graph*. The voltage quantum graph should be thought of as (not necessarily simple) quantum graph whose edges are labeled by elements of a finite abelian group G. Then, the derived quantum graph is a simple quantum graph on a larger crossed product quantum set that is constructed from the voltage quantum graph in a way that generalizes the classical derived graph construction.

In this chapter we first introduce the definition of voltage quantum graphs and their derived quantum graphs, and we show that this definition extends the classical construction of Gross and Tucker. Building on the quantum isomorphism between crossed product and tensor quantum sets from Section 11.3 we show that derived quantum graphs with respect to different group actions are quantum isomorphic. Finally, we observe that the four loopfree undirected quantum graphs of Gromada and Matsuda can be obtained as derived quantum graphs from (classical) voltage graphs.

12.1. Definition of voltage and derived quantum graphs

Throughout this section, $(\tilde{B}, \tilde{\psi})$ is a quantum set and G is a finite abelian group. Further, let $\hat{\alpha} : \hat{G} \to \mathrm{Aut}(\tilde{B}, \tilde{\psi})$ be an action of the dual group $\hat{G}$ on $(\tilde{B}, \tilde{\psi})$, and let

$$(B, \psi_{\rtimes}) := (\tilde{B}, \tilde{\psi}) \rtimes_{\hat{\alpha}} \hat{G} \quad \text{with} \quad B = \tilde{B} \rtimes_{\hat{\alpha}} \hat{G}$$

be the associated crossed product quantum set (see Definition 11.2.1). Recall the linear map $E_{\rtimes} : B \to \tilde{B}$ from Notation 11.2.2 as well as the unitary elements $(u_\chi)_{\chi \in \hat{G}} \subset B$ from Notation 11.1.1.

Definition 12.1.1. Let $(\tilde{B}, \tilde{\psi})$, $\hat{\alpha} : \hat{G} \to \mathrm{Aut}(\tilde{B}, \tilde{\psi})$, $(B, \psi_{\rtimes})$ and $(u_\chi)_{\chi \in \hat{G}} \subset B$ be as above.

(a) For every $g \in G$ let $X_g := \frac{1}{n} \sum_{\chi \in \hat{G}} \overline{\chi(g)} u_\chi u_\chi^\dagger$, i.e.

$$X_g : B \to B, \quad b \mapsto \frac{1}{n} \sum_{\chi \in \hat{G}} \overline{\chi(g)} \langle u_\chi, b \rangle_{\psi_{\rtimes}} u_\chi.$$

(b) A $(G, \hat{\alpha})$-*voltage quantum graph* (or simply *voltage quantum graph*) on the quantum set $(\tilde{B}, \tilde{\psi})$ is a family $\tilde{A} = (\tilde{A}_g)_{g \in G}$ of quantum adjacency matrices that is indexed by G and satisfies

$$\tilde{A}_g \hat{\alpha}_\chi = \hat{\alpha}_\chi \tilde{A}_g \tag{12.1}$$

for all $g \in G$ and $\chi \in \hat{G}$.

(c) The *derived quantum graph* $\tilde{A} \rtimes_{\hat{\alpha}} \hat{G} : B \to B$ on $(B, \psi_\rtimes)$ is then given by

$$\tilde{A} \rtimes_{\hat{\alpha}} \hat{G} := \sum_{g \in G} m(X_g \otimes E_\rtimes^* \tilde{A}_g E_\rtimes) m^*,$$

where we write (here and in the rest of the section) m for the multiplication map on B, i.e. $m = m_\rtimes$.

The rest of this section is devoted to proving that this definition is well defined. Before that, let us discuss the case of the trivial action in more detail.

Example 12.1.2. Let $\mathrm{triv} : \hat{G} \to \mathrm{Aut}(\tilde{B}, \tilde{\psi})$ be the trivial action on $(\tilde{B}, \tilde{\psi})$, i.e.

$$\mathrm{triv}_\chi = \mathrm{Id}_{\tilde{B}} \quad \text{for all } \chi \in \hat{G}.$$

Then condition (12.1) is satisfied by any family of quantum adjacency matrices $(\tilde{A}_g)_{g \in G}$ on $(\tilde{B}, \tilde{\psi})$. Further, on the C*-algebra side the crossed product is isomorphic to the tensor product, i.e.

$$B = \tilde{B} \rtimes_{\mathrm{triv}} \hat{G} \cong \tilde{B} \otimes C(G).$$

Recall from Example 8.2.4 that classical graphs on a finite set V correspond to quantum graphs on the quantum set $(\mathbb{C}^V, \psi_V)$. If $(\tilde{B}, \tilde{\psi})$ is a quantum set of this form, then a (G, triv)-voltage quantum graph on $(\tilde{B}, \tilde{\psi})$ can be identified with a classical voltage graph in the sense of Gross and Tucker. We will discuss this in more detail in the next section.

Even more is true: A classical G-voltage graph on a finite set V is a $(G, \hat{\alpha})$-voltage quantum graph for all actions $\hat{\alpha}$ of $\hat{G}$ on (Γ, λ). If one considers (Γ, λ) as a $(G, \hat{\alpha})$-voltage quantum graph where the action $\hat{\alpha}$ is non-trivial, then the derived quantum graph $(\Gamma, \lambda) \rtimes_{\hat{\alpha}} \hat{G}$ is a non-classical quantum graph on the crossed product quantum set $(\mathbb{C}^V, \psi_V) \rtimes_{\hat{\alpha}} \hat{G}$. In Corollary 12.4.2 we will show that this derived quantum graph is quantum isomorphic to the classical derived graph of Gross and Tucker.

Remark 12.1.3. The above definitions have a very visual interpretation which we will discuss later in more detail for the case where $(\tilde{B}, \tilde{\psi}) = (\mathbb{C}^V, \psi_V)$ is a classical set and $\hat{\alpha}$ is the trivial action. For now, let us only remark that the operation

$$(X_g, E_\rtimes^* \tilde{A}_g E_\rtimes) \mapsto m(X_g \otimes E_\rtimes^* \tilde{A}_g E_\rtimes) m^* \tag{12.2}$$

can be interpreted as taking the intersection of the two quantum graphs X_g and $E_\rtimes^* \tilde{A}_g E_\rtimes$. If $(\tilde{B}, \tilde{\psi})$ and $(B, \psi_\rtimes)$ are both classical sets and both X_g and $E_\rtimes^* \tilde{A}_g E_\rtimes$ are classical graphs, then this operation always produces a new graph (or rather adjacency matrix) which has an edge between two vertices whenever both X_g and $E_\rtimes^* \tilde{A}_g E_\rtimes$ contain this particular edge. However, in the truly quantum setting it is generally not guaranteed that two quantum adjacency matrices produces a new quantum adjacency matrix via (12.2). This is similar and related to the fact that, in a noncommutative C*-algebra, the product of two projections is not necessarily again a projection.

We proceed to prove that the maps X_g and A from the previous definition are indeed quantum adjacency matrices on the quantum set $(B, \psi_{\rtimes})$.

Lemma 12.1.4. *We have for every $g \in G$ and $bu_\chi \in B$,*

$$X_g(bu_\chi) = \overline{\chi(g)}\tilde{\psi}(b)u_\chi, \tag{12.3}$$
$$m(X_g \otimes E^*_{\rtimes}\tilde{A}_g E_{\rtimes})m^*(bu_\chi) = \overline{\chi(g)}\tilde{A}_g(b)u_\chi, \tag{12.4}$$
$$m(X_g \otimes E^*_{\rtimes}\tilde{A}_g E_{\rtimes})m^* = m(E^*_{\rtimes}\tilde{A}_g E_{\rtimes} \otimes X_g)m^*. \tag{12.5}$$

Proof. For the first equation, the definition of X_g yields

$$X_g(bu_\chi) = \frac{1}{n}\sum_{\xi \in \hat{G}} \overline{\xi(g)} u_\xi \, u_\xi^\dagger(bu_\chi),$$

where $u_\xi^\dagger(bu_\chi) = \langle u_\xi, bu_\chi\rangle_{\psi_{\rtimes}} = n\delta_{\xi=\chi}\tilde{\psi}(b)$ by (11.3). Hence, the sum over ξ collapses, and we get

$$X_g(bu_\chi) = \overline{\chi(g)}\tilde{\psi}(b)u_\chi.$$

Next, let us use the formula for m^* from (11.4) in Theorem 11.2.3 to compute the left-hand side of (12.4). For any $b \in B$ and $\chi \in \hat{G}$ we get

$$\begin{aligned} m(X_g \otimes E^*_{\rtimes}\tilde{A}_g E_{\rtimes})m^*(bu_\chi) &= \frac{1}{n}\sum_{\xi \in \hat{G}} X_g(b_i^{(1)}u_\xi)E^*_{\rtimes}\tilde{A}_g E_{\rtimes}(\hat{\alpha}_{\xi^{-1}}(b_i^{(2)})u_{\xi^{-1}\chi}), \\ &= \frac{1}{n}\sum_{\xi \in \hat{G}} \overline{\xi(g)}\tilde{\psi}(b_i^{(1)})u_\xi \, E^*_{\rtimes}\tilde{A}_g E_{\rtimes}(\hat{\alpha}_{\xi^{-1}}(b_i^{(2)})u_{\xi^{-1}\chi}). \end{aligned}$$

where $\tilde{m}^*(b) = \sum_i b_i^{(1)} \otimes b_i^{(2)}$ and we treat the sum over i as implicit. For the second step we use (12.3) from above. The definition of $E_{\rtimes}$ and $E^*_{\rtimes}$ from Notation 11.2.2 and Lemma 11.2.3 yield

$$E^*_{\rtimes}\tilde{A}_g E_{\rtimes}(\hat{\alpha}_{\xi^{-1}}(b_i^{(2)})u_{\xi^{-1}\chi}) = n\delta_{\xi=\chi}\tilde{A}_g(\hat{\alpha}_{\chi^{-1}}(b_i^{(2)})).$$

Thus, the sum over ξ collapses, and we are left with

$$m(X_g \otimes E^*_{\rtimes}\tilde{A}_g E_{\rtimes})m^*(bu_\chi) = \overline{\chi(g)}u_\chi \, \tilde{\psi}(b_i^{(1)}) \, \tilde{A}_g(\hat{\alpha}_{\chi^{-1}}(b_i^{(2)})).$$

Since by assumption $\tilde{A}_g$ is invariant under the group action $\hat{\alpha}$, this expression is equal to

$$\overline{\chi(g)}\tilde{\psi}(b_i^{(1)})\tilde{A}_g(b_i^{(2)})u_\chi = \overline{\chi(g)}\tilde{A}_g(b)u_\chi,$$

for it is one of the first properties of the comultiplication that $\psi(b_i^{(1)})T(b_i^{(2)}) = T(b)$ holds for any linear map $T : \tilde{B} \to \tilde{B}$, see Proposition 8.1.3.

To complete the proof it suffices to show that $m(E_{\rtimes}^*\tilde{A}_g E_{\rtimes} \otimes X_g)m^*$ can be expressed via the same formula, i.e.

$$m(E_{\rtimes}^*\tilde{A}_g E_{\rtimes} \otimes X_g)m^*(bu_\chi) = \frac{1}{n}\overline{\chi(g)}\tilde{A}_g(b)u_\chi.$$

This can be done with the exact same steps as above where one only needs to interchange the roles of X_g and $E_{\rtimes}^*\tilde{A}_g E_{\rtimes}$. □

Proposition 12.1.5. *The maps $X_g : B \to B$ and $A : B \to B$ from the previous definition are quantum adjacency matrices on the quantum set $(B, \psi_{\rtimes})$.*

Proof. Let us first fix some $g \in G$ and prove that X_g is a quantum adjacency matrix. For any $\chi, \xi \in \hat{G}$ one has

$$(u_\chi^\dagger \otimes u_\xi^\dagger)m^* = (m(u_\chi \otimes u_\xi))^\dagger = u_{\chi\xi}^\dagger.$$

Therefore, we obtain

$$\begin{aligned} m(X_g \otimes X_g)m^* &= \frac{1}{n^2}\sum_{\chi,\xi\in\hat{G}} \overline{\chi(g)}\,\overline{\xi(g)}\, m\left((u_\chi \otimes u_\chi^\dagger) \otimes (u_\xi \otimes u_\xi^\dagger)\right) m^* \\ &= \frac{1}{n^2}\sum_{\chi,\xi\in\hat{G}} \overline{\chi(g)}\,\overline{\xi(g)}\, u_{\chi\xi} \otimes u_{\chi\xi}^\dagger \\ &= X_g, \end{aligned}$$

and this shows that X_g is Schur-idempotent. What's more, it is an easy observation that a very similar calculation yields

$$m(X_g \otimes X_h)m^* = 0$$

if $g \neq h$. It remains to show that X_g is $*$-preserving. For any $b \in B$ and $\chi \in \hat{G}$ one first observes

$$u_\chi^\dagger(b^*) = \langle u_\chi, b^* \rangle_{\psi_{\rtimes}} = \overline{\langle u_\chi^*, b \rangle_{\psi_{\rtimes}}} = \overline{u_{\chi^{-1}}^\dagger(b)}.$$

This yields

$$\begin{aligned} X_g(b^*) &= \frac{1}{n}\sum_{\chi\in\hat{G}} \overline{\chi(g)} u_\chi\, u_\chi^\dagger(b^*) = \frac{1}{n}\sum_{\chi\in\hat{G}} \chi^{-1}(g) u_{\chi^{-1}}^*\, \overline{u_{\chi^{-1}}^\dagger(b)} \\ &= \left(\frac{1}{n}\sum_{\chi\in\hat{G}} \overline{\chi^{-1}(g)} u_{\chi^{-1}}\, u_{\chi^{-1}}^\dagger(b)\right)^* = X_g(b)^*, \end{aligned}$$

which is exactly the desired property. Thus, X_g is a quantum adjacency matrix.

Finally, let us prove that A is a quantum adjacency matrix. Using the (co)associativity of m (and m^*) as well as Lemma 12.1.4, we obtain

$$\begin{aligned} m(A \otimes A)m^* &= \frac{1}{n^2} \sum_{g,h \in G} m(m(X_g \otimes X_h)m^* \otimes m(E_{\rtimes}^* \tilde{A}_g E_{\rtimes} \otimes E_{\rtimes}^* \tilde{A}_h E_{\rtimes}))m^* \\ &= \frac{1}{n^2} \sum_{g \in G} m(X_g \otimes m(E_{\rtimes}^* \tilde{A}_g E_{\rtimes} \otimes E_{\rtimes}^* \tilde{A}_g E_{\rtimes})m^*)m^*, \end{aligned}$$

since $m(X_g \otimes X_h)m^* = \delta_{g=h} X_g$. It is not hard to see $m(E_{\rtimes}^* \otimes E_{\rtimes}^*) = E_{\rtimes}^* m$. Further, one has

$$(E_{\rtimes} \otimes E_{\rtimes})m^* = m^* E_{\rtimes}$$

since for all $bu_\chi \in B$ we have

$$\begin{aligned} (E_{\rtimes} \otimes E_{\rtimes})m^*(bu_\chi) &= \frac{1}{n} \sum_{\xi \in \hat{G}} E_{\rtimes}(b_i^{(1)} u_\xi) \otimes E_{\rtimes}(\hat{\alpha}_{\xi^{-1}}(b_i^{(2)}) u_{\xi^{-1}\chi}) \\ &= n\delta_{\chi=1} b_i^{(1)} \otimes b_i^{(2)} \\ &= n\delta_{\chi=1} m^*(b) \\ &= m^* E_{\rtimes}(bu_\chi). \end{aligned}$$

Thus, we have

$$m(E_{\rtimes}^* \tilde{A}_g E_{\rtimes} \otimes E_{\rtimes}^* \tilde{A}_g E_{\rtimes})m^* = E_{\rtimes}^* m(\tilde{A}_g \otimes \tilde{A}_g)m^* E_{\rtimes} = E_{\rtimes}^* \tilde{A}_g E_{\rtimes},$$

where we use that $\tilde{A}_g$ is a quantum adjacency matrix. Putting everything together, it follows that A is Schur-idempotent. As both the X_g and the maps $E_{\rtimes}^* \tilde{A}_g E_{\rtimes}$ are $*$-preserving, one obtains that A is $*$-preserving as well. □

12.2. Properties of the derived quantum graph

In this section we study three properties of the derived quantum graph $\tilde{A} \rtimes_{\hat{\alpha}} \hat{G}$ relative to properties of the voltage quantum graph $\tilde{A}$. These properties are: being loopfree, being undirected and being d-regular for some $d \geq 0$. Recall from Definition 8.2.5 that a quantum graph A on a quantum set (B, ψ) is called

1. *loopfree* if $m(A \otimes \mathrm{Id})m^* = 0$,
2. *undirected* if $A = A^*$ where the adjoint is taken with respect to the inner product induced by ψ, and
3. *d-regular* for some $d \geq 0$ if $A(1) = d$.

We say that a voltage quantum graph $\tilde{A} = (\tilde{A}_g)_{g\in G}$ has these properties if the same equations hold for $\sum_{g\in G} \tilde{A}_g$, respectively.

Though $\sum_{g\in G} \tilde{A}_g$ is not a quantum adjacency matrix it makes sense to think of this matrix as the adjacency matrix of a quantum multigraph. We don't make this notion precise here but refer to the discussion in [41, Section 2]. For more information on regular quantum graphs we refer to [61].

The following proposition relates properties of a derived quantum graph $\tilde{A} \rtimes_{\hat{\alpha}} \hat{G}$ to properties of the underlying voltage quantum graph $\tilde{A}$.

Proposition 12.2.1. *Let $\tilde{A} = (\tilde{A}_g)_{g\in G}$ be a $(G, \hat{\alpha})$-voltage quantum graph on a quantum set $(\tilde{B}, \tilde{\psi})$. Then the derived quantum graph $\tilde{A} \rtimes_{\hat{\alpha}} \hat{G}$ is*

1. *loopfree if and only if $\tilde{A}_1$ is loopfree,*
2. *undirected if and only if $\tilde{A}_g^* = \tilde{A}_{g^{-1}}$ holds for all $g \in G$, and*
3. *d-regular if and only if $\tilde{A}$ is d-regular, i.e. if $\sum_{g\in G} \tilde{A}_g(1) = d$.*

Proof. Let $A := \tilde{A} \rtimes_{\hat{\alpha}} \hat{G}$.

Ad (1). Using the definition of A together with co-associativity of m and (12.5) we get

$$m(A \otimes \mathrm{Id})m^* = \sum_{g\in G} m(E_{\rtimes}^* \tilde{A}_g E_{\rtimes} \otimes m(X_g \otimes \mathrm{Id})m^*)m^*.$$

A simple calculation using the comultiplication formula (11.4) reveals for $b \in \tilde{B}$ and $\chi \in \hat{G}$ that

$$\begin{aligned} m(X_g \otimes \mathrm{Id})m^*(bu_\chi) &= \frac{1}{n} \sum_{\zeta\in\hat{G}} X_g(b_i^{(1)} u_\zeta)\hat{\alpha}_{\zeta^{-1}}(b_i^{(2)})u_{\zeta^{-1}\chi} \\ &= \frac{1}{n} \sum_{\zeta\in\hat{G}} \overline{\zeta(g)}\tilde{\psi}(b_i^{(1)})b_i^{(2)}u_\chi = \delta_{g=1} bu_\chi. \end{aligned}$$

Thus, the above expression becomes

$$m(A \otimes \mathrm{Id})m^* = m(E_{\rtimes}^* \tilde{A}_1 E_{\rtimes} \otimes \mathrm{Id})m^*.$$

Using again formula (11.4) one gets in a similar way

$$\begin{aligned} m(E_{\rtimes}^* \tilde{A}_1 E_{\rtimes} \otimes \mathrm{Id})m^*(bu_\chi) &= \tilde{A}_1(b_i^{(1)})b_i^{(2)}u_\chi \\ &= \tilde{m}(\tilde{A}_1 \otimes \mathrm{Id})\tilde{m}^*(b)u_\chi. \end{aligned}$$

The statement follows immediately.

Ad (2). With formula (12.4) we have

$$A(bu_\chi) = \sum_{g\in G} \overline{\chi(g)}\tilde{A}_g(b)u_\chi$$

for all $b \in \tilde{B}$ and $\chi \in \hat{G}$. With (11.3) we see

$$\langle A(bu_\chi), cu_\xi \rangle_{\psi_\rtimes} = \sum_{g \in G} n\delta_{\chi=\xi} \langle \overline{\chi(g)} \tilde{A}_g(b), c \rangle_{\tilde{\psi}} = \sum_{g \in G} n\delta_{\chi=\xi} \langle b, \xi(g) \tilde{A}_g^*(c) \rangle_{\tilde{\psi}}$$

for all $c \in \tilde{B}$ and $\xi \in \hat{G}$, and it follows that

$$A^*(cu_\xi) = \sum_{g \in G} \xi(g) \tilde{A}_g^*(c) u_\xi = \sum_{g \in G} \overline{\xi(g^{-1})} \tilde{A}_g^*(c) u_\xi.$$

If $\tilde{A}_g^* = \tilde{A}_{g^{-1}}$ holds for all $g \in G$, then one obtains immediately that $A = A^*$. Conversely, if $A = A^*$, then we have with a substitution on the right-hand side

$$\sum_{g \in G} \overline{\chi(g)} \tilde{A}_g(b) = \sum_{g \in G} \overline{\chi(g)} \tilde{A}_{g^{-1}}^*(b)$$

for all $b \in \tilde{B}$ and $\chi \in \hat{G}$. By standard arguments, the families $(\tilde{A}_g(b))_{g \in G}$ and $(\tilde{A}_{g^{-1}}^*(b))_{g \in G}$ are equal, i.e. $\tilde{A}_g^* = \tilde{A}_{g^{-1}}$ holds for all $g \in G$.

Ad (3). Using (12.4) from Lemma 12.1.4 it is a simple observation that

$$A(1) = \sum_{g \in G} \tilde{A}_g(1) = d$$

holds if and only if $\tilde{A}$ is d-regular. □

12.3. Connection to Gross and Tucker's construction

Let us now compare our definition of voltage quantum graphs and their derived quantum graphs with Gross and Tucker's classical theory. By Example 8.2.4 a classical (simple, directed) graph with vertex set V corresponds to a quantum graph on the quantum set $(\mathbb{C}^V, \psi_V)$. We will see that in the same vein, (G, triv)-voltage quantum graphs on $(\mathbb{C}^V, \psi_V)$ correspond to classical voltage graphs, and the derived quantum graph corresponds to Gross and Tucker's derived graph.

Recall the definition of classical voltage and derived graphs from Definition 10.0.1. In full generality, neither the voltage nor the derived graph need to be simple. However, our definition of a voltage quantum graph has been designed so that the derived quantum graph is always simple in the sense that the derived quantum graph is given by a single quantum adjacency matrix. This is intended to reduce technical complications and because one is mostly interested in simple quantum graphs. Therefore, our construction can only recover the classical situation when the derived graph is simple. In Proposition 10.0.3 we characterized voltage graphs with simple derived graphs. Let us call a voltage graph that satisfies the condition from Proposition 10.0.3 *pre-simple*.

Proposition 12.3.1. *Let G be a finite abelian group and $(\mathbb{C}^V, \psi_V)$ the quantum set from Example 8.2.4 for some finite set V.*

Given a pre-simple voltage graph (Γ, λ) *on* V *with labeling* $\lambda : E \to G$*, let* $\tilde{A}_g : \mathbb{C}^V \to \mathbb{C}^V$ *be the adjacency matrix of the (simple, directed) subgraph of* Γ *that contains all edges from* Γ *that are labeled by* g*. Then the map*

$$(\Gamma, \lambda) \mapsto \tilde{A} = (\tilde{A}_g)_{g \in G}$$

is a one-to-one correspondence between G*-labeled pre-simple voltage graphs on* V *and* (G, triv)*-voltage quantum graphs on* $(\mathbb{C}^V, \psi_V)$*.*

Proof. As observed in Example 12.1.2 a (G, triv)-voltage quantum graph is nothing but a family $(\tilde{A}_g)_{g \in G}$ of quantum adjacency matrices on some quantum set. Quantum adjacency matrices on $(\mathbb{C}^V, \psi_V)$ are the same as adjacency matrices of (simple, directed) graphs on V by Example 8.2.4. Thus, it is evident that every pre-simple classical voltage graph can be interpreted as a voltage quantum graph on $(\mathbb{C}^V, \psi_V)$ as in the statement, and vice versa. □

Lemma 12.3.2. *Let* $(\mathbb{C}^V, \psi_V)$ *be the quantum set from Example 8.2.4. Further, let* G *be a finite abelian group with the trivial action* $\mathrm{triv} : \hat{G} \to \mathrm{Aut}(\mathbb{C}^V, \psi_V)$*. The following are true:*

1. *The assignment from the generators of* $\mathbb{C}^V \rtimes_{\mathrm{triv}} \hat{G}$ *into* $C(V \times G)$ *given by*

$$e_v \mapsto \sum_{g \in G} e_{v,g}, \qquad u_\chi \mapsto \sum_{v \in V, g \in G} \chi(g) e_{v,g}$$

   extends to an isomorphism

$$(\mathbb{C}^V, \psi_V) \rtimes_{\mathrm{triv}} \hat{G} \cong (\mathbb{C}^{V \times G}, \psi_{V \times G}),$$

   where the quantum set on the right-hand side is associated to the classical finite set $V \times G$ *as in Example 8.2.4, and we write* e_v*,* $e_{v,g}$ *for indicator functions in* $C(V)$ *and* $C(V \times G)$*, respectively.*

2. *The quantum graph* $X_g : B \to B$ *with* $g \in G$ *from Definition 12.1.1 corresponds to the classical graph on* $V \times G$ *that contains all edges of the form*

$$(v, h) \to (w, hg) \qquad \text{for } v, w \in V \text{ and } h \in G.$$

3. *The quantum graph* $E_\rtimes^* \tilde{A}_g E_\rtimes : B \to B$ *from Definition 12.1.1 corresponds to the classical graph on* $V \times G$ *that contains all edges of the form*

$$(v, h) \to (w, k) \qquad \text{for } v \xrightarrow{\tilde{A}_g} w \in V \text{ and } h, k \in G,$$

   where $v \xrightarrow{\tilde{A}_g} w$ *indicates that there is an edge from* v *to* w *in the classical graph corresponding to the adjacency matrix* $\tilde{A}_g$*.*

4. *Let A_1, A_2 be two quantum adjacency matrices which correspond to classical graphs Γ_1, Γ_2 on $V \times G$ in the sense of Example 8.2.4. Then, the map*

$$A_3 := m(A_1 \otimes A_2)m^*$$

is again a quantum adjacency matrix, and it corresponds to the classical graph

$$\Gamma_3 := \Gamma_1 \cap \Gamma_2,$$

which is the intersection of Γ_1 and Γ_2, i.e. Γ_3 has vertex set $V \times G$ and its edge set is the intersection of the edge sets of Γ_1 and Γ_2.

Proof. For the first statement, it suffices to recall from Example 11.2.5 that

$$(\mathbb{C}^V, \psi_V) \rtimes_{\mathrm{triv}} \hat{G} \cong (\mathbb{C}^V \otimes C(G), \psi_V \otimes \psi_G),$$

where an isomorphism is given by

$$e_v u_\chi \mapsto \sum_{g \in G} \chi(g) e_{v,g}.$$

For (b), let us identify u_χ with $\sum_{g \in G} \chi(g) e_{v,g}$ according to the above isomorphism. Then, one has for all $\chi \in \hat{G}, v \in V$ and $h \in G$

$$u_\chi^\dagger(e_{v,h}) = \langle u_\chi, e_{v,h} \rangle_{\psi_{V \otimes G}} = \left\langle \sum_{w \in V, k \in G} \chi(k) e_{w,k}, e_{v,h} \right\rangle_{\psi_{V \otimes G}} = \overline{\chi(h)}.$$

It follows

$$X_g(e_{v,h}) = \frac{1}{n} \sum_{\chi \in \hat{G}} \overline{\chi(g)} u_\chi \, u_\chi^\dagger(e_{v,h}) = \frac{1}{n} \sum_{\chi \in \hat{G}} \sum_{w \in V, k \in G} \overline{\chi(gh)} \chi(k) e_{w,k} = \sum_{w \in V} e_{w,hg},$$

where we use in the last step that $\sum_{\chi \in \hat{G}} \chi(h^{-1} g^{-1} k) = n$ if $k = hg$ and 0 otherwise. The claim follows directly.

Next, let us prove (3). A short calculation suffices to verify

$$E_\rtimes(e_v \otimes e_h) = e_v, \quad \text{and} \quad E_\rtimes^*(e_v) = e_v \otimes 1_{C(G)} = \sum_{k \in G} e_v \otimes e_k,$$

where $E_\rtimes$ and $E_\rtimes^*$ are as in Notation 11.2.2. It follows

$$E_\rtimes^* \tilde{A}_g E_\rtimes(e_v \otimes e_h) = \tilde{A}_g(e_v) \otimes 1_{C(G)} = \sum_{w \sim_{\tilde{A}_g} v} \sum_{k \in G} e_w \otimes e_k,$$

and this concludes the proof.

Finally, (4) follows immediately from the fact that the operation $m(\cdot \otimes \cdot)m^*$ on the quantum set $(\mathbb{C}^V, \psi_V)$ is the entrywise (or Schur) product of matrices, see Example 8.2.4. Indeed, if A_1, A_2 are the adjacency matrices of classical graphs Γ_1, Γ_2 on V, then the entrywise product $A_3 = m(A_1 \otimes A_2)m^*$ of A_1 and A_2 is exactly the adjacency matrix of the intersection graph $\Gamma_1 \cap \Gamma_2$. $\square$

Now, we are ready to see that our derived quantum graph construction generalizes the classical construction from pre-simple voltage graphs.

Proposition 12.3.3. *In the situation of Proposition 12.3.1 one has for a pre-simple voltage graph* (Γ, λ) *and the associated* (G, triv)*-voltage quantum graph* $\tilde{A} = (\tilde{A}_g)_{g\in G}$ *the identification*

$$\Gamma^\lambda = \tilde{A} \rtimes_{\mathrm{triv}} \hat{G}$$

of the derived graphs, where Γ^λ *is Gross and Tucker's derived graph,* $\tilde{A} \rtimes_{\mathrm{triv}} \hat{G}$ *is the derived quantum graph from Definition 12.1.1, and we identify classical graphs on* $V \times G$ *with quantum graphs on* $(\mathbb{C}^V, \psi_V) \rtimes_{\mathrm{triv}} \hat{G}$ *according to Example 8.2.4.*

Proof. Evidently, the derived quantum graph $\tilde{A} \rtimes_{\mathrm{triv}} \hat{G}$ is defined on the quantum set $(\mathbb{C}^V, \psi_V) \rtimes_{\mathrm{triv}} \hat{G}$, and by Lemma 12.3.2 (a) this quantum set is isomorphic to $(C(V \times G), \psi_{V\times G})$. Thus, the derived quantum graph corresponds to some classical graph on the set $V \times G$ (due to Example 8.2.4). It remains to check that this classical graph is exactly Gross and Tucker's derived graph Γ^λ.

Recall from Definition 12.1.1 that

$$\tilde{A} \rtimes_{\mathrm{triv}} \hat{G} = \sum_{g\in G} m\left(X_g \otimes E_{\rtimes}^* \tilde{A}_g E_{\rtimes}\right) m^*.$$

In view of Lemma 12.3.2, for any fixed $g \in G$ the map

$$m\left(X_g \otimes E_{\rtimes}^* \tilde{A}_g E_{\rtimes}\right) m^*$$

corresponds to the classical graph on $V \times G$ that contains all edges of the form

$$(v, h) \to (w, hg) \qquad \text{for } v \xrightarrow{\tilde{A}_g} w \in V \text{ and } h \in G.$$

It follows immediately that the derived quantum graph $\tilde{A} \rtimes_{\mathrm{triv}} \hat{G}$ corresponds to the classical graph on $V \times G$ that contains all edges of the form

$$(v, h) \to (w, h\lambda(e)) \qquad \text{for } e : v \to w \in \Gamma \text{ and } h \in G,$$

and this is exactly Gross and Tucker's derived graph Γ^λ. □

Finally, let us illustrate the previous proposition on a concrete example.

Example 12.3.4. Consider the graph Γ with one vertex v and one loop e based at v. Let us label the loop with the generator $1_{\mathbb{Z}_3}$ of the group $\mathbb{Z}_3 = \{0_{\mathbb{Z}_3}, 1_{\mathbb{Z}_3}, 2_{\mathbb{Z}_3}\}$. Thus, we have a voltage graph (Γ, λ) with $\lambda(e) = 1_{\mathbb{Z}_3}$. Then, the derived graph Γ^λ has three vertices v_0, v_1, v_2 corresponding to the elements of $\mathbb{Z}_3$ and three edges as sketched in Figure 12.1.

As discussed above, a classical graph on n vertices corresponds to a quantum graph on a quantum set $(\mathbb{C}^n, \psi_n)$. The voltage graph (Γ, λ) is identified with a (G, triv)-voltage

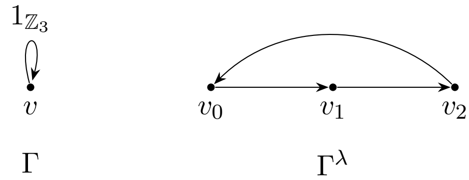


Figure 12.1.: The voltage graph (Γ, λ) and its derived graph Γ^λ.

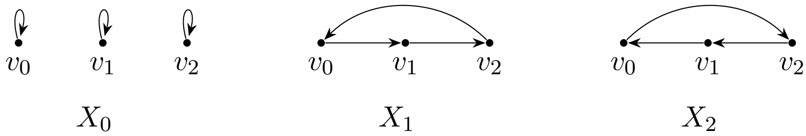


Figure 12.2.: The graphs corresponding to the adjacency matrices X_g.

quantum graph on $(\mathbb{C}, \psi_1)$ which is given by a family $(\tilde{A}_g)_{g\in\mathbb{Z}_3}$ of quantum adjacency matrices on $(\mathbb{C}, \psi_1)$. Since Γ has only one edge which is labeled by $1_{\mathbb{Z}_3}$, it is not hard to check that (Γ, λ) corresponds to the family $\tilde{A} = (\tilde{A}_g)_{g\in\mathbb{Z}_3}$ given by

$$\tilde{A}_{0_{\mathbb{Z}_3}} = 0, \quad \tilde{A}_{1_{\mathbb{Z}_3}} = \mathrm{Id}_{\mathbb{C}}, \quad \tilde{A}_{2_{\mathbb{Z}_3}} = 0.$$

Generally, we would have to check that $(\tilde{A}_g)_{g\in\mathbb{Z}_3}$ satisfies the conditions from Definition 12.1.1 to be a voltage quantum graph. However, in this case we consider $(\tilde{A}_g)_{g\in\mathbb{Z}_3}$ as a voltage quantum graph with respect to the trivial action triv of $\hat{\mathbb{Z}}_3$ on $\mathbb{C}$; thus, there is nothing to check (see Example 12.1.2).

What is the derived quantum graph $\tilde{A} \rtimes_{\mathrm{triv}} \hat{\mathbb{Z}}_3$ of this voltage quantum graph? First, recall that the derived quantum graph is defined on the crossed product quantum set

$$(\mathbb{C}, \psi_1) \rtimes_{\mathrm{triv}} \hat{\mathbb{Z}}_3.$$

By Lemma 12.3.2 (a) this quantum set is isomorphic to $(\mathbb{C}^3, \psi_3)$, and quantum graphs on $(\mathbb{C}^3, \psi_3)$ correspond to classical graph on three vertices.

Next, by Definition 12.1.1 the derived quantum graph $A := \tilde{A} \rtimes_{\mathrm{triv}} \hat{\mathbb{Z}}_3 : \mathbb{C}^3 \to \mathbb{C}^3$ on $(\mathbb{C}^3, \psi_3)$ is given by the formula

$$A = \sum_{g\in\mathbb{Z}_3} m(X_g \otimes E_{\rtimes}^* \tilde{A}_g E_{\rtimes}) m^*.$$

According to Lemma 12.3.2 (b) and (c) the quantum graphs X_g and $E_{\rtimes}^* \tilde{A}_g E_{\rtimes}$ correspond to the classical graphs on the set $\{v_0, v_1, v_2\}$ in Figure 12.2 and Figure 12.3, respectively.

By Lemma 12.3.2 (d) the term

$$m(X_g \otimes E_{\rtimes}^* \tilde{A}_g E_{\rtimes}) m^*$$

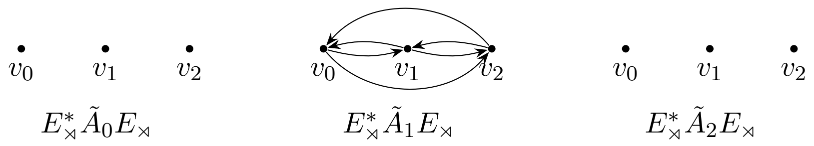


Figure 12.3.: The graphs corresponding to the adjacency matrices $E^*_{\rtimes}\tilde{A}_g E_{\rtimes}$.

returns the adjacency matrix of the classical graph that is obtained by taking the intersection of the graphs from Figure 12.2 and Figure 12.3. Thus, only the term for $g = 1_{\mathbb{Z}_3}$ contributes to the derived quantum graph A, and we have

$$A = m(X_1 \otimes E^*_{\rtimes}\tilde{A}_1 E_{\rtimes})m^*.$$

It is not hard to see that this graph is exactly the derived graph Γ^λ from Figure 12.1.

12.4. Quantum isomorphisms between derived quantum graphs

In Example 11.2.5 we discussed the special case of a crossed product quantum set with respect to the trivial action. In this case, one has an isomorphism of quantum sets

$$(\tilde{B}, \tilde{\psi}) \rtimes_{\mathrm{triv}} \hat{G} \cong (\tilde{B}, \tilde{\psi}) \otimes (C(G), \psi_G)$$

between the crossed product quantum set and the tensor product quantum set. More generally, we have seen in Proposition 11.3.1 that for any action $\hat{\alpha}$ of $\hat{G}$ on $(\tilde{B}, \tilde{\psi})$ there is a quantum isomorphism

$$(\tilde{B}, \tilde{\psi}) \rtimes_{\hat{\alpha}} \hat{G} \cong_q (\tilde{B}, \tilde{\psi}) \otimes (C(G), \psi_G)$$

between the crossed product quantum set and the tensor product quantum set. This quantum isomorphism is given by the map

$$\rho : \begin{cases} \tilde{B} \otimes C(G) \to \left(\tilde{B} \rtimes_{\hat{\alpha}} \hat{G}\right) \otimes B(\mathcal{H}), \\ \quad b \otimes \tilde{u}_\chi \mapsto \displaystyle\sum_{\zeta \in \hat{G}} u_\zeta b u_\zeta^* u_\chi \otimes E_{\zeta, \chi^{-1}\zeta}, \end{cases}$$

where $\mathcal{H} = L^2(G)$. In this section, we show that ρ is not only a quantum isomorphism between quantum sets but also a quantum isomorphism between the derived quantum graphs

$$\tilde{A} \rtimes_{\hat{\alpha}} \hat{G} \quad \text{and} \quad \tilde{A} \rtimes_{\mathrm{triv}} \hat{G}.$$

Throughout the following, we use the same notation as in Section 11.3.

Theorem 12.4.1. *Let $(\tilde{B}, \tilde{\psi})$, $\hat{\alpha} : \hat{G} \to \mathrm{Aut}(\tilde{B}, \tilde{\psi})$ and $(\tilde{B} \rtimes_{\hat{\alpha}} \hat{G}, \psi_{\rtimes})$ be as above. Further, let $\tilde{A} = (\tilde{A}_g)_{g \in G}$ be a $(G, \hat{\alpha})$-voltage quantum graph on $(\tilde{B}, \tilde{\psi})$. Then we have a quantum isomorphism*

$$\tilde{A} \rtimes_{\hat{\alpha}} \hat{G} \cong_q \tilde{A} \rtimes_{\mathrm{triv}} G$$

which is given by the map $\rho : \tilde{B} \otimes C(G) \to (\tilde{B} \rtimes_{\hat{\alpha}} \hat{G}) \otimes B(\mathcal{H})$ from Proposition 11.3.1.

Proof. We need to show that

$$(\tilde{A} \rtimes_{\hat{\alpha}} \hat{G})\rho = \rho(\tilde{A} \rtimes_{\mathrm{triv}} \hat{G})$$

holds where the left-hand side is to be interpreted as in Notation 8.3.3. First, recall

$$\begin{aligned} \tilde{A} \rtimes_{\hat{\alpha}} \hat{G} &= \sum_{g \in G} m_{\rtimes}(X_g^{\rtimes} \otimes E_{\rtimes}^* \tilde{A}_g E_{\rtimes}) m_{\rtimes}^* \\ \tilde{A} \rtimes_{\mathrm{triv}} \hat{G} &= \sum_{g \in G} m_{\otimes}(X_g^{\otimes} \otimes E_{\otimes}^* \tilde{A}_g E_{\otimes}) m_{\otimes}^*, \end{aligned}$$

where we distinguish the structure maps m, E, X_g on the crossed product quantum sets $\tilde{B} \rtimes_{\hat{\alpha}} \hat{G}$ and $\tilde{B} \otimes C(G) \cong \tilde{B} \rtimes_{\mathrm{triv}} \hat{G}$ by super-/subscripts $\rtimes$ and $\otimes$, respectively. We will show for any fixed $g \in G$ that

$$(m_{\rtimes}(X_g^{\rtimes} \otimes E_{\rtimes}^* \tilde{A}_g E_{\rtimes}) m_{\rtimes}^*)\rho = \rho(m_{\otimes}(X_g^{\otimes} \otimes E_{\otimes}^* \tilde{A}_g E_{\otimes}) m_{\otimes}^*)$$

holds. From Proposition 8.3.4 we know the equalities

$$\rho m_{\otimes} = m_{\rtimes}(\rho \boxtimes \rho) \quad \text{and} \quad m_{\rtimes}^* \rho = (\rho \boxtimes \rho) m_{\otimes}^*.$$

Thus, all we need to do is to prove

$$X_g^{\rtimes} \rho = \rho X_g^{\otimes} \quad \text{and} \quad E_{\rtimes}^* \tilde{A}_g E_{\rtimes} \rho = \rho(E_{\otimes}^* \tilde{A}_g E_{\otimes}).$$

To show the first equality we need to consider for $b \in \tilde{B}$ and $\xi \in \hat{G}$,

$$X_g^{\rtimes} \rho(b \otimes \tilde{u}_\xi) = \frac{1}{n} \sum_{\chi \in \hat{G}} \overline{\chi(g)} u_\chi \psi_{\rtimes}(u_\chi^* \rho(b \otimes \tilde{u}_\xi)).$$

For fixed $\chi \in \hat{G}$ the last factor can be computed as

$$\begin{aligned} \psi_{\rtimes}(u_\chi^* \rho(b \otimes \tilde{u}_\xi)) &= \sum_{\zeta \in \hat{G}} \psi_{\rtimes}(u_\chi^* u_\zeta b u_\zeta^* u_\xi) E_{\zeta, \chi^{-1}\zeta} \\ &= \sum_{\zeta \in \hat{G}} n \delta_{\chi = \xi} \tilde{\psi}(b) E_{\zeta, \chi^{-1}\zeta} \\ &= n \delta_{\chi = \xi} \tilde{\psi}(b) \hat{u}_\chi, \end{aligned}$$

where $\hat{u}_\chi \in B(L^2(G))$ is the operator from the proof of Proposition 11.3.1 given by $\hat{u}_\chi e_\zeta = e_{\chi\zeta}$ for all $\zeta \in \hat{G}$. It follows

$$X_g^{\rtimes} \rho(b \otimes \tilde{u}_\xi) = \overline{\xi(g)}\tilde{\psi}(b) u_\xi \otimes \hat{u}_\xi. \tag{12.6}$$

On the other hand, we compute

$$\begin{aligned}
\rho X_g^{\otimes}(b \otimes \tilde{u}_\xi) &= \rho\left(\frac{1}{n}\sum_{\chi\in\hat{G}} \overline{\chi(g)}(1\otimes \tilde{u}_\chi)(\tilde{\psi}\otimes\psi_G)(b\otimes \tilde{u}_\chi^*\tilde{u}_\xi)\right) \\
&= \rho\left(\overline{\xi(g)}\tilde{\psi}(b)(1\otimes \tilde{u}_\xi)\right) \\
&= \overline{\xi(g)}\tilde{\psi}(b) u_\xi \otimes \hat{u}_\xi.
\end{aligned} \tag{12.7}$$

Comparing (12.6) and (12.7) shows the first desired equality.

It remains to prove the second equality, i.e. $E_{\rtimes}^*\tilde{A}_g E_{\rtimes}\rho = \rho(E_{\otimes}^*\tilde{A}_g E_{\otimes})$. Let us consider again fixed $b \in \tilde{B}$ and $\xi \in \hat{G}$. We have on the one hand

$$\begin{aligned}
E_{\rtimes}^*\tilde{A}_g E_{\rtimes}\rho(b\otimes\tilde{u}_\xi) &= \sum_{\zeta\in\hat{G}} E_{\rtimes}^*\tilde{A}_g E_{\rtimes}(u_\zeta b u_\zeta^* u_\xi \otimes E_{\zeta,\xi^{-1}\zeta}) \\
&= n\delta_{\xi=1}\sum_{\zeta\in\hat{G}} \tilde{A}_g(u_\zeta b u_\zeta^*)\otimes E_{\zeta,\zeta} \\
&= n\delta_{\xi=1}\sum_{\zeta\in\hat{G}} u_\zeta\tilde{A}_g(b)u_\zeta^*\otimes E_{\zeta,\zeta},
\end{aligned} \tag{12.8}$$

(where we use in final step that $\tilde{A}_g$ commutes with the action $\hat{\alpha}$), and on the other hand

$$\begin{aligned}
\rho(E_{\otimes}^*\tilde{A}_g E_{\otimes})(b\otimes\tilde{u}_\xi) &= \rho\left(n\delta_{\xi=1}\tilde{A}_g(b)\otimes\tilde{u}_1\right) \\
&= n\delta_{\xi=1}\sum_{\zeta\in\hat{G}} u_\zeta\tilde{A}_g(b)u_\zeta^*\otimes E_{\zeta,\zeta}.
\end{aligned} \tag{12.9}$$

A comparison of (12.8) and (12.9) finishes the proof. □

If the derived quantum graph $\tilde{A}\rtimes_{\mathrm{triv}}\hat{G}$ from Theorem 12.4.1 is a classical graph, then the previous theorem yields a method to produce quantum versions of this given classical graph, i.e. quantum graphs that are quantum isomorphic to the original classical graph.

Corollary 12.4.2. *Assume that Γ^λ is a classical derived graph of a G-voltage graph (Γ,λ) on a vertex set V for a finite abelian group G. Then for any action $\hat{\alpha}:\hat{G}\to \mathrm{Aut}(\Gamma,\lambda)$ of the dual group on the voltage graph we obtain a quantum isomorphism*

$$\Gamma^\lambda \cong_q (\Gamma,\lambda)\rtimes_{\hat{\alpha}}\hat{G},$$

where we consider (Γ,λ) on the right-hand side as a $(G,\hat{\alpha})$-voltage quantum graph on $(\mathbb{C}^V,\psi_V)$ as in Example 12.1.2.

By an action of $\hat{G}$ on (Γ,λ) we mean an action $\hat{\alpha}:\hat{G}\to\mathrm{Aut}(V)$ which leaves the set of g-colored edges in Γ invariant for all $g\in G$.

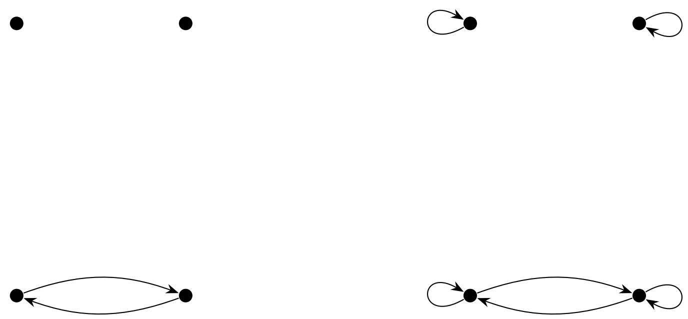


Figure 12.4.: All directed graphs on two vertices that are invariant under swapping the vertices.

12.5. Quantum graphs on M_2 as derived quantum graphs

Can we obtain the quantum graphs on $(M_2, 2\mathrm{Tr})$ from Section 9.3 as derived quantum graphs? For that, we would first need to describe $(M_2, 2\mathrm{Tr})$ as a crossed product quantum set. It is not hard to see that one has

$$(M_2, 2\mathrm{Tr}) \cong (\mathbb{C}^2, \psi_2) \rtimes_{\hat{\alpha}} \hat{\mathbb{Z}}_2,$$

where $\hat{Z}_2 = \{\hat{0}, \hat{1}\}$ is the dual group of $\mathbb{Z}_2$ and $\hat{\alpha}$ is the action of $\hat{\mathbb{Z}}_2$ on $\mathbb{C}^2$ which swaps the two elements of $\mathbb{C}^2$, i.e.

$$\hat{\alpha}_{\hat{0}} = \mathrm{Id}_{\mathbb{C}^2} \quad \text{and} \quad \hat{\alpha}_{\hat{1}}(x, y) = (y, x) \quad \text{for all } (x, y) \in \mathbb{C}^2.$$

Now, a voltage quantum graph on $(\mathbb{C}^2, \psi_2)$ with respect to this action $\hat{\alpha}$ is given by two adjacency matrices $\tilde{A}_0$ and $\tilde{A}_1$ on $(\mathbb{C}^2, \psi_2)$ which are invariant under swapping the two elements of $\mathbb{C}^2$. There are exactly four such directed graphs, depicted in Figure 12.4. It follows that there are exactly 16 distinct $(\mathbb{Z}_2, \hat{\alpha})$-voltage quantum graphs $\tilde{A} = (\tilde{A}_0, \tilde{A}_1)$ on $(\mathbb{C}^2, \psi_2)$.

All these voltage quantum graphs $\tilde{A}$ are undirected since the graphs in Figure 12.4 are undirected (in the sense that for every edge they also contain the inverse edge); therefore, by Proposition 12.2.1 the corresponding derived quantum graph $\tilde{A} \rtimes_{\hat{\alpha}} \hat{Z}_2$ will also be undirected. Further, the same proposition tells us that the derived quantum graph is loopfree if and only if $\tilde{A}_0$ has no loops. Moreover, $\tilde{A} \rtimes_{\hat{\alpha}} \hat{Z}_2$ is d-regular if and only if $\tilde{A}$ is d-regular. Thus, we arrive at the following proposition.

Proposition 12.5.1. *Every loopfree undirected quantum graph on* $(M_2, 2\mathrm{Tr})$ *is isomorphic to the derived quantum graph of a voltage quantum graph on* $(\mathbb{C}^2, \psi_2)$ *with respect to the action* $\hat{\alpha}$ *of* $\hat{\mathbb{Z}}_2$.

Proof. Recall that there are up to isomorphism exactly four loopfree undirected quantum graphs on $(M_2, 2\mathrm{Tr})$ which are given by quantum adjacency matrices A_0, A_1, A_2, A_3 such that A_d is d-regular for $d = 0, 1, 2, 3$. On the other hand, the previous argument shows

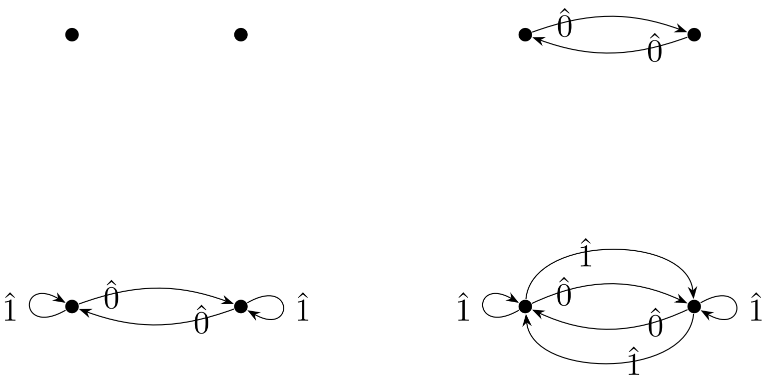


Figure 12.5.: The four voltage quantum graphs $\tilde{A}^{(0)}, \tilde{A}^{(1)}, \tilde{A}^{(2)}, \tilde{A}^{(3)}$ on $(\mathbb{C}^2, \psi_2)$ (clockwise from top left).

that any $(\mathbb{Z}_2, \hat{\alpha})$-voltage quantum graph $\tilde{A}$ on $(\mathbb{C}^2, \psi_2)$ is given by a pair $(\tilde{A}_0, \tilde{A}_1)$ of graphs from Figure 12.4. The derived quantum graph $\tilde{A} \rtimes_{\hat{\alpha}} \hat{Z}_2$ is always undirected, and it is loopfree if $\tilde{A}_0$ has no loops. Further, it is d-regular if and only if $\tilde{A}$ is d-regular. It is not hard to check that $\tilde{A}$ is d-regular if and only if the graphs $\tilde{A}_0$ and $\tilde{A}_1$ together have $2d$ edges. Therefore, we have $A_d \cong \tilde{A}^{(d)} \rtimes_{\hat{\alpha}} \hat{Z}_2$ for the four voltage quantum graphs $\tilde{A}^{(d)}$ on $(\mathbb{C}^2, \psi_2)$ depicted in Figure 12.5. $\square$

Remark 12.5.2.

1. The previous proposition can be combined with Corollary 12.4.1 to find that the loopfree quantum graphs are quantum isomorphic to classical graphs on four vertices. The respective classical graphs are obtained as the classical derived graphs of the voltage graphs in Figure 12.5. This reproves the result from Matsuda which we repeated in Theorem 9.3.2.

2. The above proposition is very specific for quantum graphs on $(M_2, 2\mathrm{Tr})$ and cannot be generalized to higher dimensions. For instance, if all undirected loopfree quantum graphs on $(M_3, 3\mathrm{Tr})$ were derived quantum graphs of voltage quantum graphs on $(\mathbb{C}^3, \psi_3)$, then they would all be quantum isomorphic to classical graphs. However, Gromada presented an example of a quantum graph on $(M_3, 3\mathrm{Tr})$ that is not quantum isomorphic to any classical graph [41, Example 8.5].

13. Quantum Gross–Tucker Theorem

Which quantum graphs are derived quantum graphs of a voltage quantum graph as in Definition 12.1.1? In the classical situation, Gross and Tucker proved that whenever a graph Γ admits a free action $\alpha : G \to \mathrm{Aut}(\Gamma)$ of a finite group G, then there is a suitable G-labeling λ of the quotient graph Γ/G such that

$$\Gamma \cong (\Gamma/G)^{\lambda},$$

i.e. Γ is the derived graph of the voltage graph $(\Gamma/G, \lambda)$ [42, Theorem 2.2.2]. In this section, we prove a quantum version of this result.

Thus, the setting is the following: We have given a quantum graph $A : B \to B$ on a quantum set (B, ψ) together with an action $\alpha : G \to \mathrm{Aut}(A)$ of a finite abelian group G on the quantum graph A. This means that α is an action of G on the C*-algebra B which satisfies

$$\psi = \psi\alpha_g \quad \text{and} \quad \alpha_g A = A\alpha_g$$

for all $g \in G$.

Where Gross and Tucker asked that the action on the graph be free, we assume that the action α satisfies the assumptions of Landstad's theorem including the additional condition for the quantum set version from Proposition 11.4.2. Thus, we ask that there is a unitary representation $(u_\chi)_{\chi\in\hat{G}} \subset B$ of the dual group $\hat{G}$ inside B such that

$$\alpha_g(u_\chi) = \chi(g)u_\chi \quad \text{and} \quad \psi = \psi\mathrm{Ad}(u_\chi)$$

hold for all $g \in G$ and $\chi \in \hat{G}$. Furthermore, also the quantum graph A must be invariant under $\mathrm{Ad}(u_\chi)$, i.e. we will also require

$$\mathrm{Ad}(u_\chi)A = A\mathrm{Ad}(u_\chi). \tag{13.1}$$

Then, in view of Proposition 11.4.2 the quantum set (B, ψ) is a crossed product quantum set

$$(B, \psi) = (B^\alpha, \tilde{\psi}) \rtimes_{\hat{\alpha}} \hat{G}.$$

In particular,

- $B = B^\alpha \rtimes_{\hat{\alpha}} \hat{G}$,
- every element of B can be written uniquely as $b = \sum_{\chi\in\hat{G}} b_\chi u_\chi$ for suitable $b_\chi \in B^\alpha$,

- $\psi(b_\chi u_\chi) = \delta_{\chi=1}\psi(b_\chi) = n\delta_{\chi=1}\tilde{\psi}(b_\chi)$ holds for all $\chi \in \hat{G}$ and $b_\chi \in B^\alpha$ with $n = |G|$.

Note that we consistently use the notation from Notation 11.2.2 for crossed product quantum sets with B^α in the place of $\tilde{B}$ as well as the notation from Landstad's theorem (Theorem 11.4.1) and Proposition 11.4.2.

Let us first justify condition (13.1) by showing that it is automatically satisfied by any derived quantum graph.

Lemma 13.0.1. *Let $\tilde{A} = (\tilde{A}_g)_{g\in G}$ be a $(G,\hat{\alpha})$-voltage quantum graph on $(\tilde{B},\tilde{\psi})$ and let $A = \tilde{A} \rtimes_{\hat{\alpha}} \hat{G}$ be the associated derived quantum graph on the crossed product quantum set $(B,\psi_\rtimes)$. Then A satisfies* (13.1)*, i.e. A is invariant under the conjugation action $(\mathrm{Ad}(u_\chi))_{\chi\in\hat{G}}$*

Proof. Recall the definition of A from Definition 12.1.1 via

$$A = \sum_{g\in G} m(X_g \otimes E_\rtimes^* \tilde{A}_g E_\rtimes)m^*.$$

Let us fix some $\chi \in \hat{G}$. Evidently we have $\mathrm{Ad}(u_\chi)m = m(\mathrm{Ad}(u_\chi)\otimes \mathrm{Ad}(u_\chi))$, and by taking the adjoint one obtains $(\mathrm{Ad}(u_\chi)\otimes\mathrm{Ad}(u_\chi))m^* = m^*\mathrm{Ad}(u_\chi)$. Here we use that the adjoint of $\mathrm{Ad}(u_\chi)$ is $\mathrm{Ad}(u_\chi^*)$, which follows from the definition using that $\psi_\rtimes$ is invariant under $\mathrm{Ad}(u_\chi)$.

Therefore, it suffices to show that both X_g and $E_\rtimes^*\tilde{A}_g E_\rtimes$ commute with $\mathrm{Ad}(u_\chi)$ for all $g \in G$. For the maps X_g this follows from

$$u_\zeta^\dagger \mathrm{Ad}(u_\chi) = \langle u_\zeta, \mathrm{Ad}(u_\chi)(\cdot)\rangle_{\psi_\rtimes} = \langle u_\chi^* u_\zeta u_\chi, \cdot\rangle_{\psi_\rtimes} = \langle u_\zeta, \cdot\rangle_{\psi_\rtimes} \quad \text{for all } \zeta \in \hat{G},$$

where we use that $\hat{G}$ is abelian.

For $E_\rtimes^*\tilde{A}_g E_\rtimes$ observe that

$$E_\rtimes \mathrm{Ad}(u_\chi)(bu_\zeta) = E_\rtimes(\hat{\alpha}_\chi(b)u_\zeta) = n\delta_{\zeta=1}\hat{\alpha}_\chi(b) = \hat{\alpha}_\chi E_\rtimes(bu_\zeta)$$

holds for all $b \in \tilde{B}$ and $\zeta \in \hat{G}$, again using that $\hat{G}$ is abelian. By taking the adjoint we also get $\mathrm{Ad}(u_\chi)E_\rtimes^* = E_\rtimes^*\hat{\alpha}_\chi$. Now, the voltage-graph condition (12.1) yields

$$\mathrm{Ad}(u_\chi)E_\rtimes^*\tilde{A}_g E_\rtimes = E_\rtimes^*\hat{\alpha}_\chi\tilde{A}_g E_\rtimes = E_\rtimes^*\tilde{A}_g\hat{\alpha}_\chi E_\rtimes = E_\rtimes^*\tilde{A}_g E_\rtimes \mathrm{Ad}(u_\chi),$$

which is all that we need. □

Theorem 13.0.2. *Let $A : B \to B$ be a quantum graph on the quantum set (B,ψ) and let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite abelian group G on A which satisfies the assumptions of Landstad's theorem for quantum sets (Proposition 11.4.2) as well as the additional assumption* (13.1)*. Then the following are true:*

1. *The family $\tilde{A} := (\tilde{A}_g)_{g\in G}$ with*

$$\tilde{A}_g = m^*(X_g \otimes A)m|_{B^\alpha}$$

is a $(G,\hat{\alpha})$-voltage quantum graph on $(B^\alpha,\tilde{\psi})$, where $X_g = \frac{1}{n}\sum_{\chi\in\hat{G}}\overline{\chi(g)}u_\chi u_\chi^\dagger$ with $n = |G|$ is the quantum adjacency matrix from Definition 12.1.1 and Proposition 12.1.5.

2. *One has*

$$A \cong \tilde{A} \rtimes_{\hat{\alpha}} \hat{G},$$

i.e. A is recovered as the derived quantum graph of $\tilde{A}$ as in Definition 12.1.1.

Remark 13.0.3. 1. The claim that $\tilde{A}$ is a voltage quantum graph entails that every single map $\tilde{A}_g$ maps into $B^\alpha \subset B$. This is not immediately clear since the maps m and m^* are defined on the larger algebra B.

2. Let us remark again that the operation

$$(A_1, A_2) \mapsto m(A_1 \otimes A_2)m^*$$

intuitively means to take the intersection of quantum graphs as discussed in Remark 12.1.3.

3. Conversely, assume that $A \cong \tilde{A} \rtimes_{\hat{\alpha}} \hat{G}$ is known. Then the underlying quantum set (B, ψ) is isomorphic to a crossed product quantum set which entails that the conditions of Landstad's theorem are satisfied. Further, the additional assumption (13.1) is automatically satisfied by Lemma 13.0.1. Thus, all the conditions in the theorem are necessary for derived quantum graphs.

4. It is possible that a given quantum graph A can be expressed as a derived quantum graph in multiple ways. Indeed, the trivial group acts on any quantum graph A and this action satisfies the assumptions of the theorem. However, this case is not very insightful since the underlying voltage quantum graph is simply A itself.

Before we prove the theorem, we will investigate some properties of the maps $\tilde{A}_g$. First, we pick up the issue mentioned in the previous remark and show that the maps $\tilde{A}_g$ are well-defined as maps from B^α into B^α.

Proposition 13.0.4. *The maps $\tilde{A}_g$ are well-defined, i.e. we have for all $g \in G$,*

$$m(X_g \otimes A)m^*(B^\alpha) \subset B^\alpha.$$

Proof. It suffices to show $m(X_g \otimes A)m^*\alpha_h = \alpha_h m(X_g \otimes A)m^*$ for all $h \in G$. Evidently, $m(\alpha_h \otimes \alpha_h) = \alpha_h m$ holds since α_h is a $*$-automorphism. Further, one readily verifies $\alpha_h^* = \alpha_{h^{-1}}$ with respect to the inner product induced by ψ. By taking the adjoint, the previous identity entails $(\alpha_h \otimes \alpha_h)m^* = m^*\alpha_h$ for all $h \in G$. Further, $A\alpha_h = \alpha_h A$ is true since $\alpha_h \in \mathrm{Aut}(A)$ by assumption.

It only remains to show $X_g\alpha_h = \alpha_h X_g$ for all $h \in G$. A quick calculation reveals

$$u_\chi^\dagger \alpha_h = \chi(h) u_\chi^\dagger$$

for all $\chi \in \hat{G}$ and $h \in G$. (Recall $u_\chi^\dagger = \psi(u_\chi^* \cdot)$). Consequently, we have

$$X_g\alpha_h = \frac{1}{n}\sum_{\chi \in \hat{G}} \overline{\chi(g)} u_\chi u_\chi^\dagger \alpha_h = \frac{1}{n}\sum_{\chi \in \hat{G}} \overline{\chi(g)}\chi(h) u_\chi u_\chi^\dagger = \alpha_h X_g,$$

since $\alpha_h(u_\chi) = \chi(h)u_\chi$ is true for all $\chi \in \hat{G}$. Altogether, we obtain the identity $m(X_g \otimes A)m^*\alpha_h = \alpha_h m(X_g \otimes A)m^*$ as desired. □

The next lemma establishes a connection between the maps $\tilde{A}_g$ and a family of maps $\tilde{A}_\chi : B^\alpha \to B^\alpha$ which look like the equivalents of the $\tilde{A}_g$ under a kind of Fourier transform. These maps have convenient algebraic properties and yield an alternative definition of the maps $\tilde{A}_g$.

Lemma 13.0.5. *There are linear maps $\tilde{A}_\chi : B^\alpha \to B^\alpha$ such that*

$$A(bu_\chi) = \tilde{A}_\chi(b)u_\chi \tag{13.2}$$

is true for all $b \in B^\alpha$ and $\chi \in \hat{G}$. Then we have for all $b \in B^\alpha$, $g \in G$ and $\chi, \zeta \in \hat{G}$:

$$\tilde{A}_\chi\hat{\alpha}_\zeta = \hat{\alpha}_\zeta\tilde{A}_\chi, \tag{13.3}$$

$$\tilde{A}_\chi(b) = \frac{1}{n}\sum_{\zeta\in\hat{G}} \tilde{A}_\zeta\left(b_i^{(1)}\right)\tilde{A}_{\zeta^{-1}\chi}\left(b_i^{(2)}\right), \tag{13.4}$$

$$\tilde{A}_\chi(b)^* = \tilde{A}_{\chi^{-1}}(b^*), \tag{13.5}$$

$$\tilde{A}_g(b) = \frac{1}{n}\sum_{\zeta\in\hat{G}} \zeta(g)\tilde{A}_\zeta(b). \tag{13.6}$$

where in the second line $\tilde{m}^*(b) = \sum_i b_i^{(1)} \otimes b_i^{(2)}$ and we sum implicitly over i.

Proof. Recall that α respects the quantum adjacency matrix A in the sense that $A\alpha_g = \alpha_g A$ holds for all $g \in G$. It follows immediately that A preserves the subalgebras $B^\alpha u_\chi$, i.e.

$$A(B^\alpha u_\chi) \subset B^\alpha u_\chi$$

holds for all $\chi \in \hat{G}$. Therefore, the maps $\tilde{A}_\chi$ from (13.2) exist.

Ad (13.3). Using the definition of $\tilde{A}_\chi$ and the equality $A\hat{\alpha}_\zeta = \hat{\alpha}_\zeta A$ one calculates for all $b \in B^\alpha$

$$\tilde{A}_\chi\left(\hat{\alpha}_\zeta(b)\right) = A(\hat{\alpha}_\zeta(b)u_\chi)u_{\chi^{-1}} = \hat{\alpha}_\zeta\left(A(bu_\chi)\right)u_{\chi^{-1}} = \hat{\alpha}_\zeta\left(\tilde{A}_\chi(b)\right).$$

Ad (13.4). Since A is a quantum adjacency matrix, we have $m(A \otimes A)m^* = A$. Using the formula (11.4) for the comultiplication on B we get for all $b \in B^\alpha, \chi \in \hat{G}$,

$$\begin{aligned}
A(bu_\chi) &= \frac{1}{n}\sum_{\zeta\in\hat{G}} A\left(b_i^{(1)}u_\zeta\right)A\left(\hat{\alpha}_{\zeta^{-1}}\left(b_i^{(2)}\right)u_{\zeta^{-1}\chi}\right) \\
&= \frac{1}{n}\sum_{\zeta\in\hat{G}} \tilde{A}_\zeta\left(b_i^{(1)}\right)\tilde{\alpha}_\zeta\left(\tilde{A}_{\zeta^{-1}\chi}\left(\hat{\alpha}_{\zeta^{-1}}\left(b_i^{(2)}\right)\right)\right)u_\zeta u_{\zeta^{-1}\chi} \\
&= \frac{1}{n}\sum_{\zeta\in\hat{G}} \tilde{A}_\zeta\left(b_i^{(1)}\right)\tilde{A}_{\zeta^{-1}\chi}\left(b_i^{(2)}\right)u_\chi,
\end{aligned}$$

where we use $\tilde{A}_{\zeta^{-1}\chi}\hat{\alpha}_{\zeta^{-1}} = \hat{\alpha}_{\zeta^{-1}}\tilde{A}_{\zeta^{-1}\chi}$ in the second step. Comparing this with (13.2) yields the desired formula.

Ad (13.5). Since A is a quantum adjacency matrix it is $*$-preserving in the sense that $A\left((bu_\chi)^*\right) = A(bu_\chi)^*$ holds for all $b \in B^\alpha$ and $\chi \in \hat{G}$. With the definition of $\tilde{A}_\chi$ it follows

$$\begin{aligned}\tilde{A}_{\chi^{-1}}(\hat{\alpha}_{\chi^{-1}}(b^*))u_{\chi^{-1}} &= A(\hat{\alpha}_{\chi^{-1}}(b^*)u_{\chi^{-1}}) = A\left((bu_\chi)^*\right) \\ &= A(bu_\chi)^* = \left(\tilde{A}_\chi(b)u_\chi\right)^* = \hat{\alpha}_{\chi^{-1}}\left(\tilde{A}_\chi(b)^*\right)u_{\chi^{-1}}.\end{aligned}$$

The claim follows by using $\tilde{A}_{\chi^{-1}}\hat{\alpha}_{\chi^{-1}} = \hat{\alpha}_{\chi^{-1}}\tilde{A}_{\chi^{-1}}$ and applying $\hat{\alpha}_\chi$ on both sides.

Ad (13.6). Let us use the definition of $\tilde{A}_g$ together with the formula (11.4) for the comultiplication on B and (12.3) to obtain

$$\begin{aligned}\tilde{A}_g(b) &= m(X_g \otimes A)m^*(b) \\ &= \frac{1}{n}\sum_{\zeta\in\hat{G}} X_g(b_i^{(1)}u_\zeta)A\left(\hat{\alpha}_{\zeta^{-1}}\left(b_i^{(2)}\right)u_{\zeta^{-1}}\right) \\ &= \frac{1}{n}\sum_{\zeta\in\hat{G}} \overline{\zeta(g)}\tilde{\psi}(b_i^{(1)})u_\zeta A\left(\hat{\alpha}_{\zeta^{-1}}\left(b_i^{(2)}\right)u_{\zeta^{-1}}\right).\end{aligned}$$

Since A commutes with the action $\hat{\alpha}$ the properties of the comultiplication on B imply that this expression is equal to

$$\frac{1}{n}\sum_{\zeta\in\hat{G}} \overline{\zeta(g)}\tilde{\psi}(b_i^{(1)})\tilde{A}_{\zeta^{-1}}\left(b_i^{(2)}\right) = \frac{1}{n}\sum_{\zeta\in\hat{G}} \overline{\zeta(g)}\tilde{A}_{\zeta^{-1}}(b).$$

The statement follows by a change of variable. □

We are now ready to prove the main theorem of this section which has been stated before as Theorem 13.0.2.

Proof of Theorem 13.0.2. We first show that each $\tilde{A}_g$ is a quantum adjacency matrix on $(B^\alpha, \tilde{\psi})$ which is invariant under the action $\hat{\alpha}$. To start with, let us verify that $\tilde{A}_g$ is Schur-idempotent. Using formula (13.6) from the previous lemma, we get for all $b \in B^\alpha$,

$$\begin{aligned}\tilde{m}\left(\tilde{A}_g \otimes \tilde{A}_g\right)\tilde{m}^*(b) &= \frac{1}{n^2}\sum_{\chi,\xi\in\hat{G}} \chi(g)\tilde{A}_\chi(b_i^{(1)})\xi(g)\tilde{A}_\xi(b_i^{(2)}) \\ &= \frac{1}{n^2}\sum_{\eta\in\hat{G}}\eta(g)\sum_{\chi,\xi:\chi\xi=\eta}\tilde{A}_\chi(b_i^{(1)})\tilde{A}_\xi(b_i^{(2)}).\end{aligned}$$

By (13.4) we have $\frac{1}{n}\sum_{\chi,\xi:\chi\xi=\eta}\tilde{A}_\chi(b_i^{(1)})\tilde{A}_\xi(b_i^{(2)}) = \tilde{A}_\eta(b)$ for all $\eta \in \hat{G}$. Thus, the above is equal to

$$\frac{1}{n}\sum_{\eta\in\hat{G}}\eta(g)\tilde{A}_\eta(b) = \tilde{A}_g(b).$$

Next, we show that $\tilde{A}_g$ is $*$-preserving. Using (13.5) and the formula (13.6) from the previous lemma one gets for all $b \in B^\alpha$ and $g \in G$,

$$\begin{aligned}\tilde{A}_g(b^*) &= \frac{1}{n}\sum_{\zeta\in\hat{G}} \zeta(g)\tilde{A}_\zeta(b^*)\\ &= \frac{1}{n}\sum_{\zeta\in\hat{G}} \zeta(g)\tilde{A}_{\zeta^{-1}}(b)^*\\ &= \left(\frac{1}{n}\sum_{\zeta\in\hat{G}} \zeta^{-1}(g)\tilde{A}_{\zeta^{-1}}(b)\right)^*\\ &= \tilde{A}_g(b)^*,\end{aligned}$$

as desired. Altogether, it follows that $\tilde{A}_g$ is a quantum adjacency matrix. For $\tilde{A}$ being a voltage graph we also need $\tilde{A}_g\hat{\alpha}_\chi = \hat{\alpha}_\chi\tilde{A}_g$ for all $\chi \in \hat{G}$. However, this follows immediately by combining (13.3) and (13.6).

It remains to prove $A = \tilde{A} \rtimes_{\hat{\alpha}} \hat{G}$, i.e.

$$A = \sum_{g\in G} m(X_g \otimes E_\rtimes^* \tilde{A}_g E_\rtimes)m^*$$

Using the formula for the comultiplication (11.4) as well as (11.5) one immediately obtains for all $b \in B^\alpha$ and $\chi \in \hat{G}$,

$$\begin{aligned}m(X_g \otimes E_\rtimes^*\tilde{A}_gE_\rtimes)m^*(bu_\chi) &= \frac{1}{n}\sum_{\zeta\in\hat{G}} X_g\left(b_i^{(1)}u_\zeta\right) n\delta_{\zeta^{-1}\chi=1}\tilde{A}_g\left(\hat{\alpha}_{\zeta^{-1}}(b_i^{(2)})\right)\\ &= X_g\left(b_i^{(1)}u_\chi\right)\tilde{A}_g\left(\hat{\alpha}_{\chi^{-1}}(b_i^{(2)})\right)\\ &= \overline{\chi(g)}\tilde{\psi}\left(b_i^{(1)}\right)\tilde{A}_g\left(b_i^{(2)}\right)u_\chi,\end{aligned}$$

where we use (12.3) in the last step. As $\left(\tilde{\psi}\otimes \mathrm{Id}_{B^\alpha}\right)\tilde{m}^* = \mathrm{Id}_{B^\alpha}$, the above reduces to

$$\overline{\chi(g)}\tilde{A}_g(b)u_\chi.$$

By summing over all $g \in G$ and using the formula (13.6) for $\tilde{A}_g$ we arrive at

$$\begin{aligned}\sum_{g\in G} m^*(X_g \otimes E_\rtimes^*\tilde{A}_gE_\rtimes)m(bu_\chi) &= \frac{1}{n}\sum_{g\in G}\sum_{\zeta\in\hat{G}} \overline{\chi(g)}\zeta(g)A_\zeta(b)u_\chi\\ &= A_\chi(b)u_\chi,\end{aligned}$$

for $\sum_{g\in G}\chi(g)\overline{\zeta(g)} = n\delta_{\zeta=\chi}$. This concludes the proof. $\square$

Remark 13.0.6. If A is in fact a classical graph (i.e. a quantum graph on $(\mathbb{C}^V, \psi_V)$ for some finite set V), then an action $\alpha : G \to \mathrm{Aut}(A)$ satisfies the conditions of Landstad's theorem if and only if it is a free action. Moreover, the notorious operators $\mathrm{Ad}(u_\chi)$ on $\mathbb{C}^V$ are trivial, and therefore the additional conditions

$$\psi_V = \psi_V \,\mathrm{Ad}(u_\chi) \quad \text{and} \quad A\,\mathrm{Ad}(u_\chi) = \mathrm{Ad}(u_\chi)A$$

for Theorem 13.0.2 are automatically satisfied. The quantum set $(B^\alpha, \tilde{\psi})$ is clearly isomorphic to $(\mathbb{C}^{V/G}, \psi_{V/G})$ and the induced action $\hat{\alpha}$ is trivial. Thus, in the classical case our Gross–Tucker theorem just says that a free action of a finite abelian group on a graph A gives rise to a voltage graph structure on the quotient graph A/G such that the derived graph is isomorphic to A. This recovers exactly the classical Gross–Tucker theorem.

Backmatter

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List of Symbols

(B, ψ)	Quantum set
(Γ, λ)	(Classical) voltage graph
$A : B \to B$	Quantum adjacency matrix
$B(\mathcal{H})$	Bounded operators on $\mathcal{H}$
B^α	Fixed point algebra
B^{op}	Opposite algebra of a C*-algebra B
$C(X), C_0(X)$	C*-Algebra of continuous functions (vanishing at infinity)
$C^*(G)$	Bipartite graph C*-algebra associated with G
$G, \hat{G}$	Finite abelian group and its dual group
$G = (U, V, E)$	Finite bipartite graph with vertex sets U, V and edge set E
I, I_n	$n \times n$ identity matrix
$K_{n,m}$	Complete bipartite graph with n and m vertices
M_n	Complex $n \times n$ matrix algebra
$Q_n(c)$	Subhypercube of Q_n induced by an edge weighting
$Q_n = (U_n, V_n, E_n)$	n-dimensional hypercube graph
$X \# S$	Action of $B \otimes B^{\mathrm{op}}$ on B given by $(b \otimes c)\#a := bac$
$\mathbb{C}$	Set of complex numbers
Δ_n	n-dimensional standard simplex
Γ^λ	(Classical) derived graph of a (classical) voltage graph
$\mathbb{N}$	Set of natural numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{Z}$	Set of integers
$\mathbb{Z}_n$	Cyclic group of order n
$\alpha, \hat{\alpha}$	Action of a (finite abelian) group G and its dual group $\hat{G}$, resp.
$\cong$	Isomorphism of C*-algebras, quantum sets or quantum graphs
$\cong_q$	Quantum isomorphism of quantum sets or quantum graphs
$\delta_{\chi=\xi}$	Kronecker delta, i.e. $\delta_{\chi=\xi} = 1$ if $\chi = \xi$ and $\delta_{\chi=\xi} = 0$ otherwise
$\mathcal{H}$	Hilbert space

$\mathrm{Aut}(A)$	Automorphism group of a quantum graph
$\mathrm{Aut}(B, \psi)$	Automorphism group of a quantum set
Id_B	Identity operator on a C*-algebra B
Tr, tr	Un-normalized and normalized trace on M_n, resp.
$\mathrm{par}_k(i)$	Parity of the first $k+1$-binary digits of i
triv	Trivial action of a group
$\oplus$	Direct sum of C*-algebras, Hilbert spaces and representations
$\otimes$	Tensor product of C*-algebras
$\psi_{\rtimes}$	Quantum set functional on a crossed product quantum set $\tilde{B} \rtimes_{\hat{\alpha}} \hat{G}$
ρ_c	Representation of a hypercube graph C*-algebra induced by an admissible edge weighting $c : E_n \to \mathbb{C}$
$\rho_{\mathbf{t}}$	Representation of a hypercube graph C*-algebra associated with $\mathbf{t} \in \Delta_{n-1}$
$\tilde{A} \rtimes_{\hat{\alpha}} \hat{G}$	Derived quantum graph of a $(G, \hat{\alpha})$-voltage quantum graph $\tilde{A}$
$\tilde{A} = (\tilde{A}_g)_{g \in G}$	$(G, \hat{\alpha})$-voltage quantum graph
$\tilde{B} \rtimes_{\hat{\alpha}} \hat{G}$	Crossed product C*-algebra
$b^{\dagger}$	The functional $\psi(b^* \cdot)$ on a C*-algebra B with respect to a quantum set functional ψ
$i\#k$	(Binary) number obtained by flipping the k-th binary digit of i
p_x	Canonical projection generator of a bipartite graph C*-algebra corresponding to a vertex x
x^*	Adjoint of an element or operator

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